The LLL Algorithm: Lattice Basis Reduction and applications to Approximate Shortest Vector Problem

Lucas Petit

May 26, 2025

Recap: Euclidean Space and Inner Product

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We consider a real finite-dimensional vector space $\mathbb{R}^n$ equipped with the standard **Euclidean inner product**:

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This inner product induces the **Euclidean norm**:

$$\|\mathbf{u}\|_2 = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{\sum_{i=1}^{n} \mathbf{u}_i^2}$$

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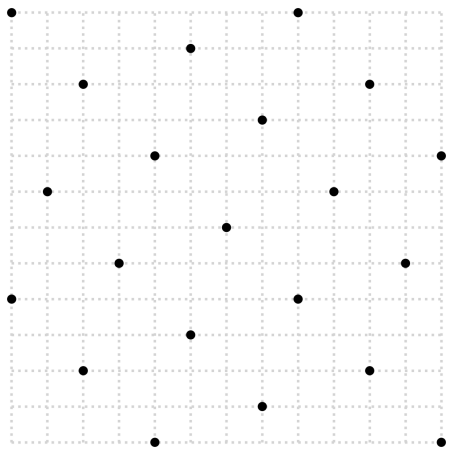

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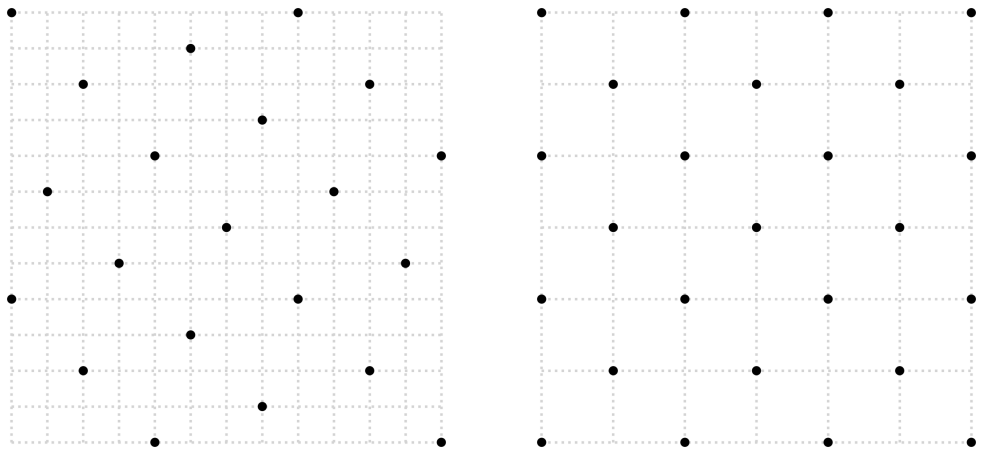

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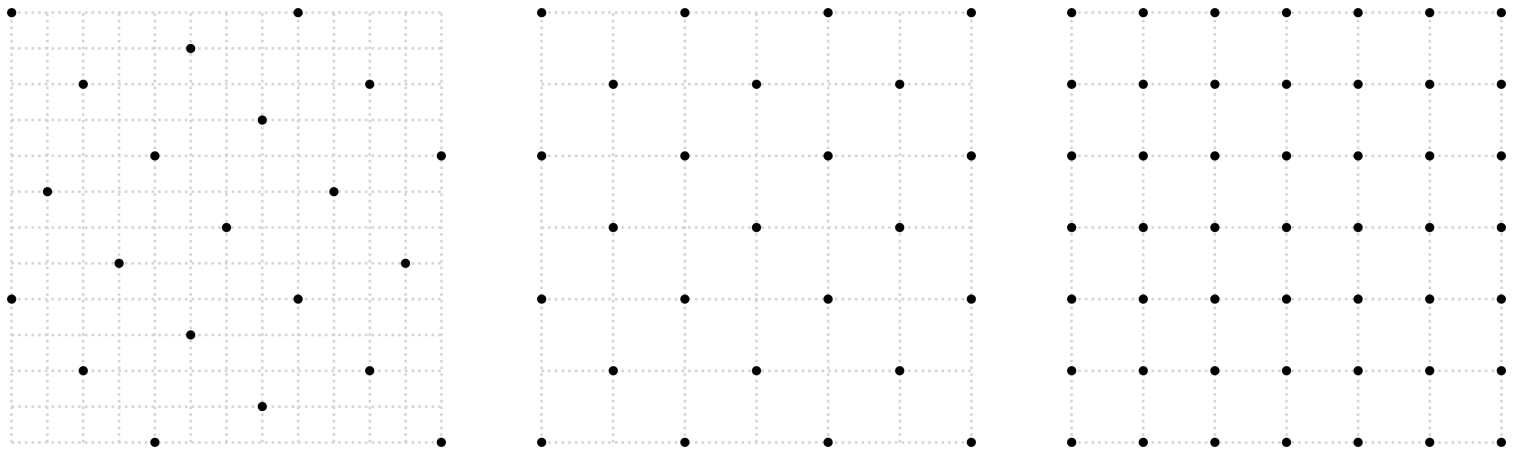

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Any lattice $\mathscr{L} \subseteq \mathbb{R}^n$ admits a maximal $\mathbb{Z}$-linearly independent family $(\mathbf{b}_i)_{1 \leq i \leq m}$, with $m \leq n$ such that:

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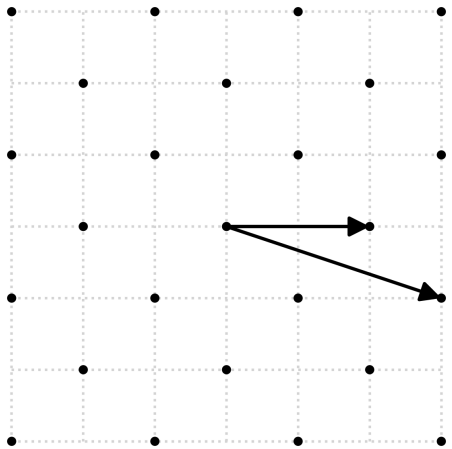

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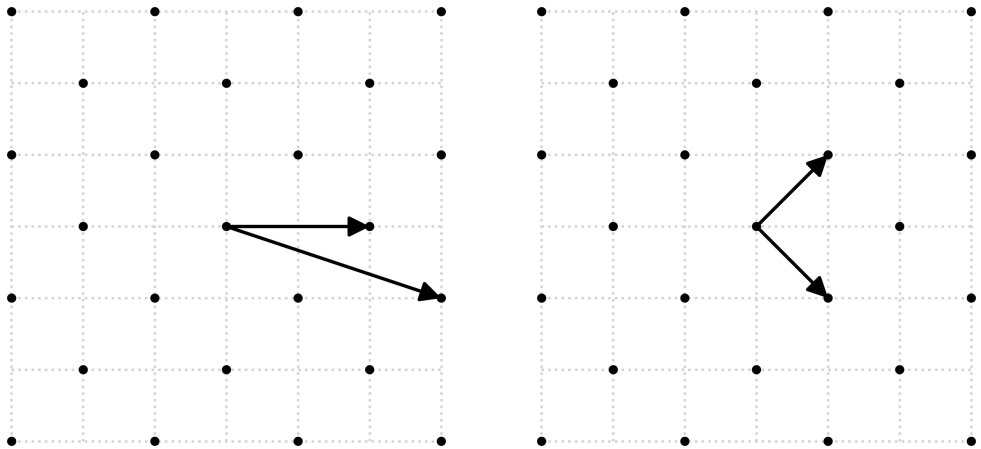

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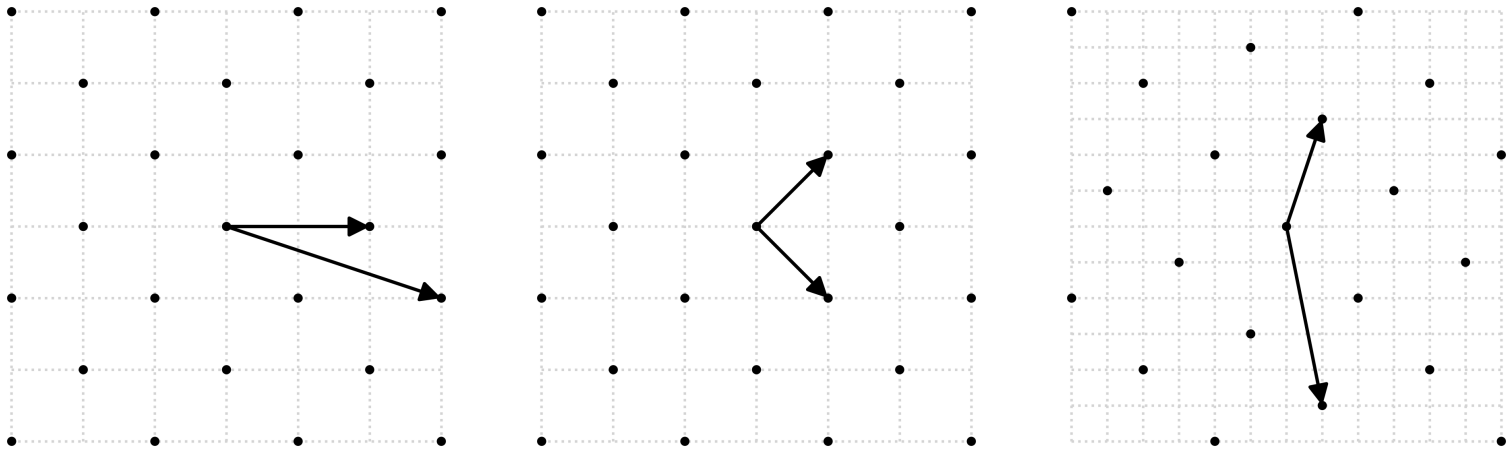

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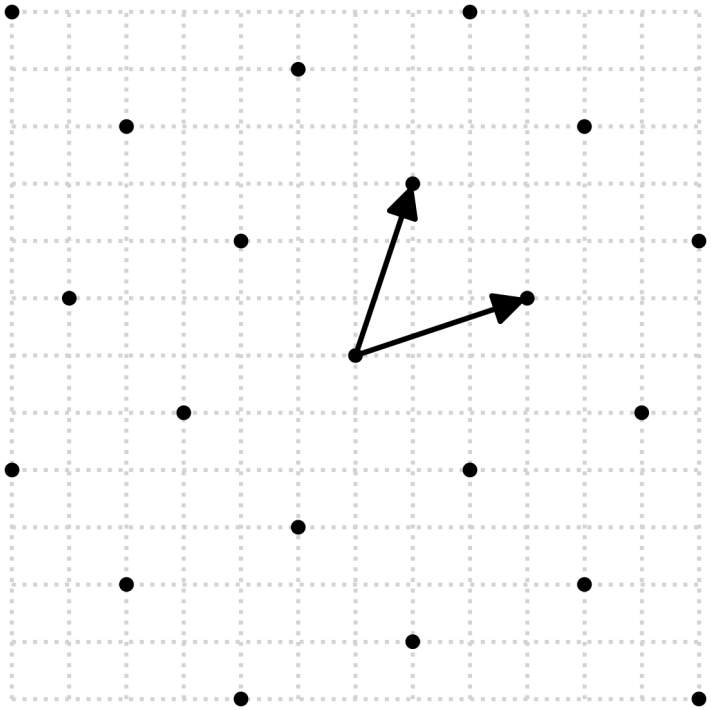

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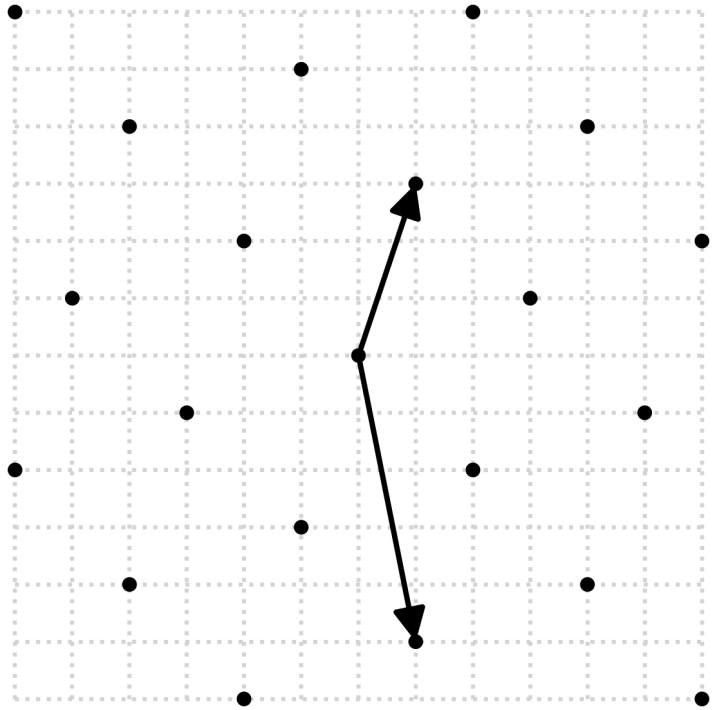

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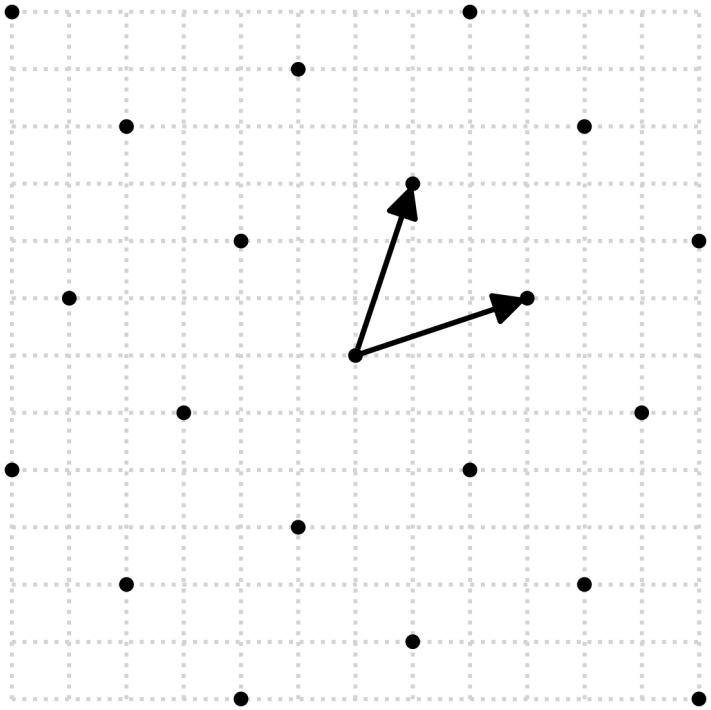

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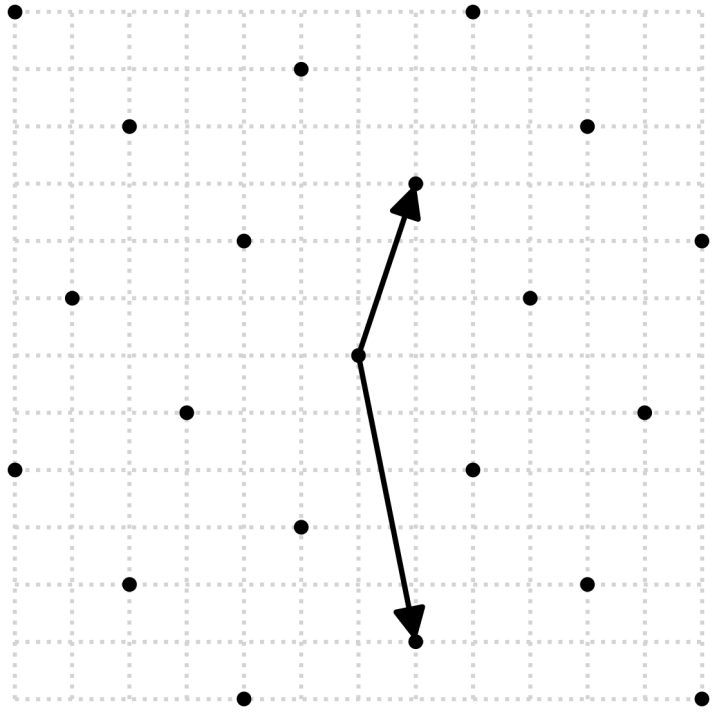

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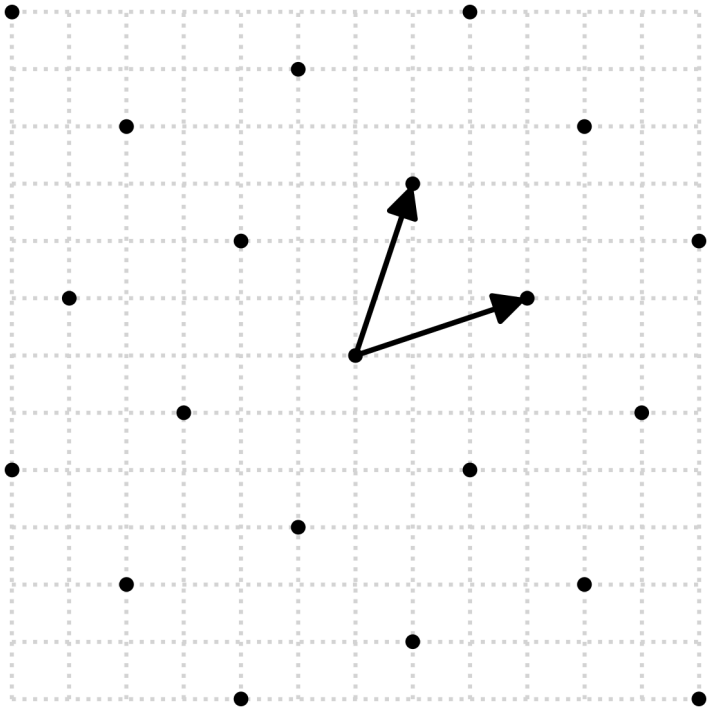

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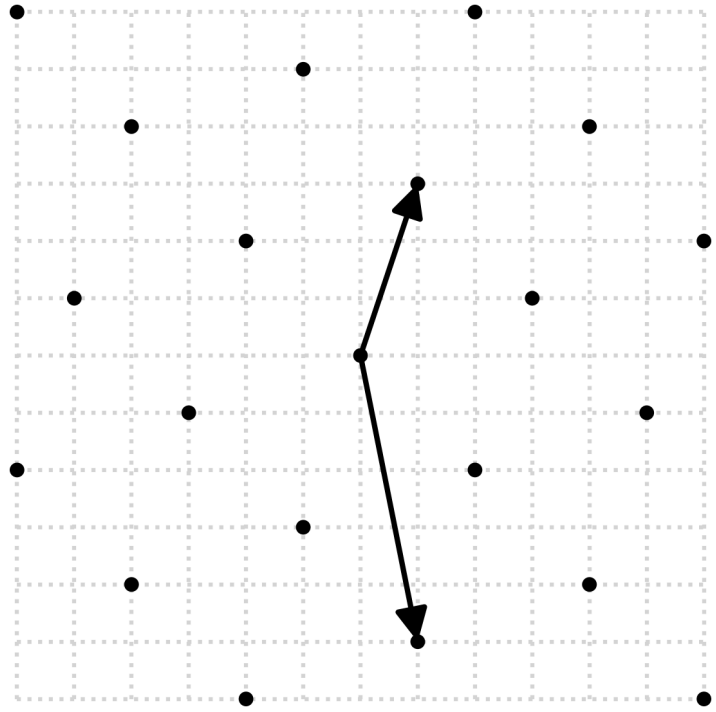

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$\rightarrow$ notion of **quasi-orthogonal** (or **reduced**) bases.

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A basis $(\mathbf{b}_i)_{1 \leq i \leq n}$ of $\mathbb{R}^n$ is called **orthogonal** if

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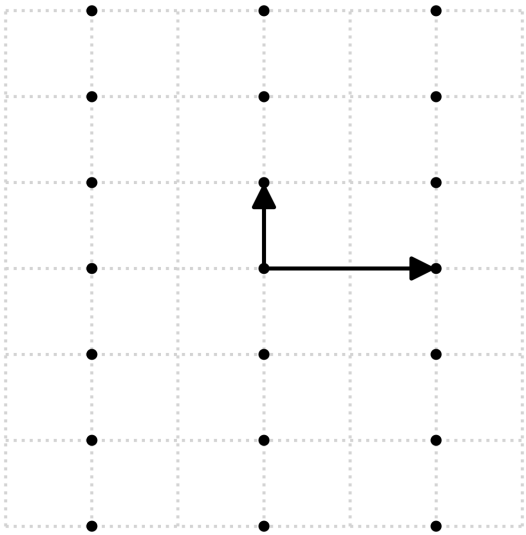

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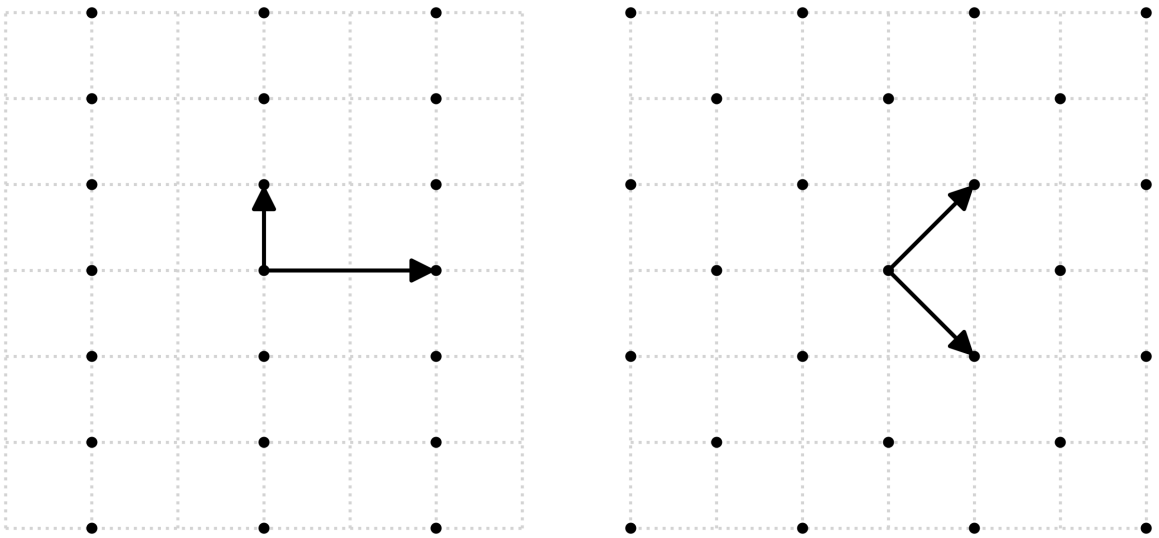

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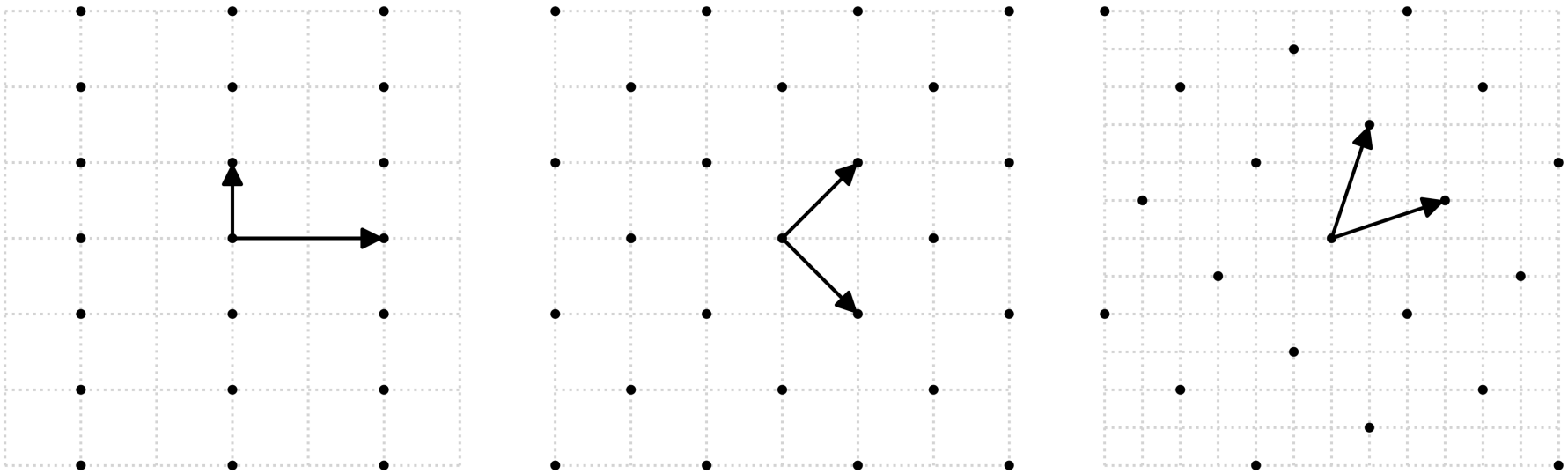

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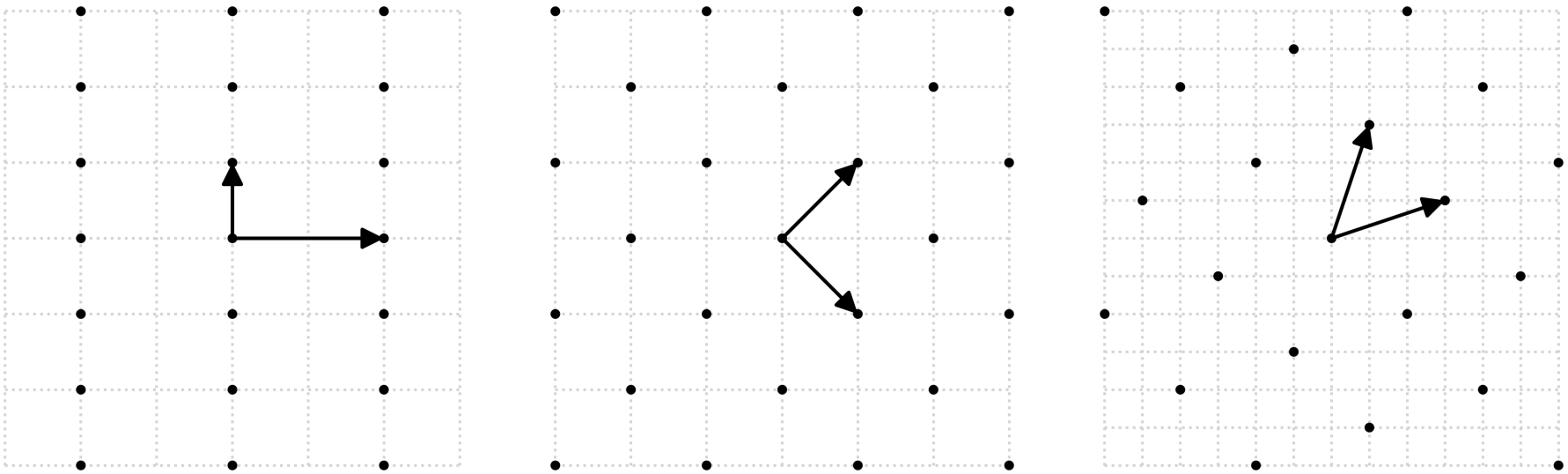

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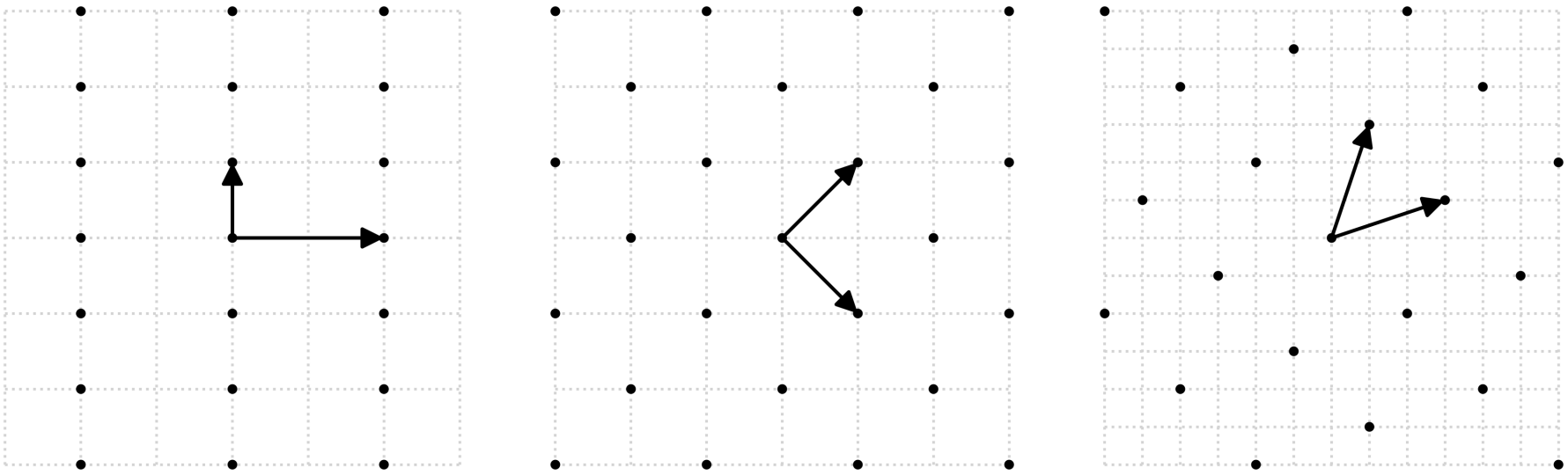

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$\rightarrow$ **Gram-Schmidt orthogonalization process**

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Let $(\mathbf{b}_i)_{1\leq i\leq n}$ be a basis of $\mathbb{R}^n$. The associated orthogonal basis $(\mathbf{b}_i^*)_{1\leq i\leq n}$ is constructed via the **Gram–Schmidt orthogonalization process**:

$$\mathbf{b}_1^* := \mathbf{b}_1, \quad \mathbf{b}_i^* := \mathbf{b}_i - \sum_{j=1}^{i-1} \mu_{i,j}\mathbf{b}_j^*, \quad \mu_{i,j} := \frac{\langle \mathbf{b}_i, \mathbf{b}_j^* \rangle}{\|\mathbf{b}_j^*\|^2}.$$

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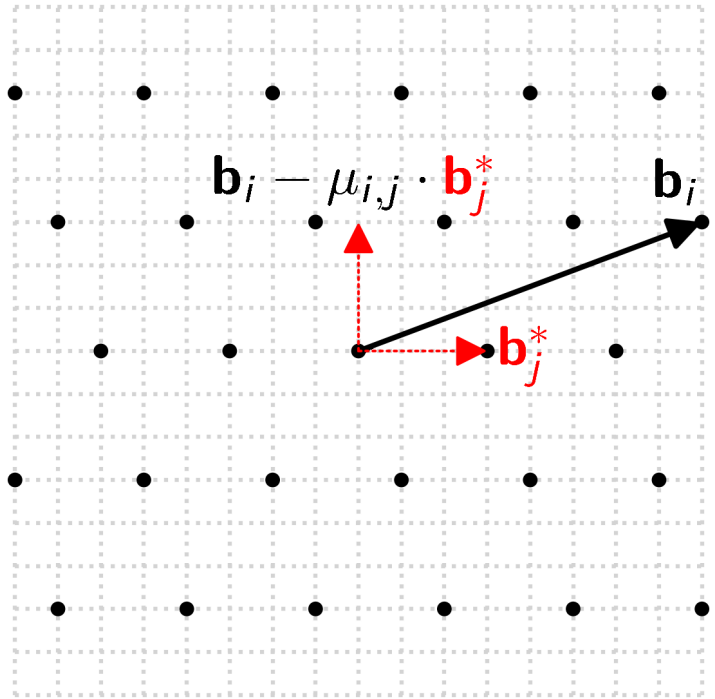

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The coefficients $\mu_{i,j}$ are called **Gram–Schmidt coefficients**.

$$\begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \mu_{2,1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \mu_{n,1} & \cdots & \mu_{n,n-1} & 1 \end{pmatrix} \times \begin{pmatrix} \mathbf{b}_1^* \\ \mathbf{b}_2^* \\ \vdots \\ \mathbf{b}_n^* \end{pmatrix}$$

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The resulting family $(\mathbf{b}_i^*)_{1 \leq i \leq n}$ is orthogonal.

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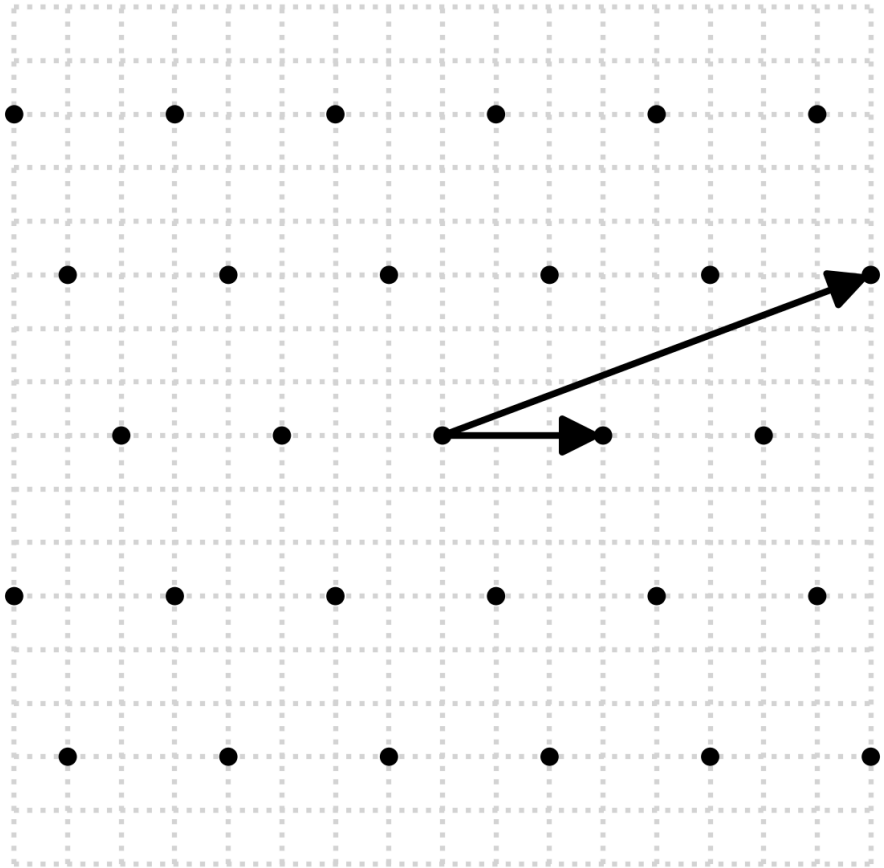

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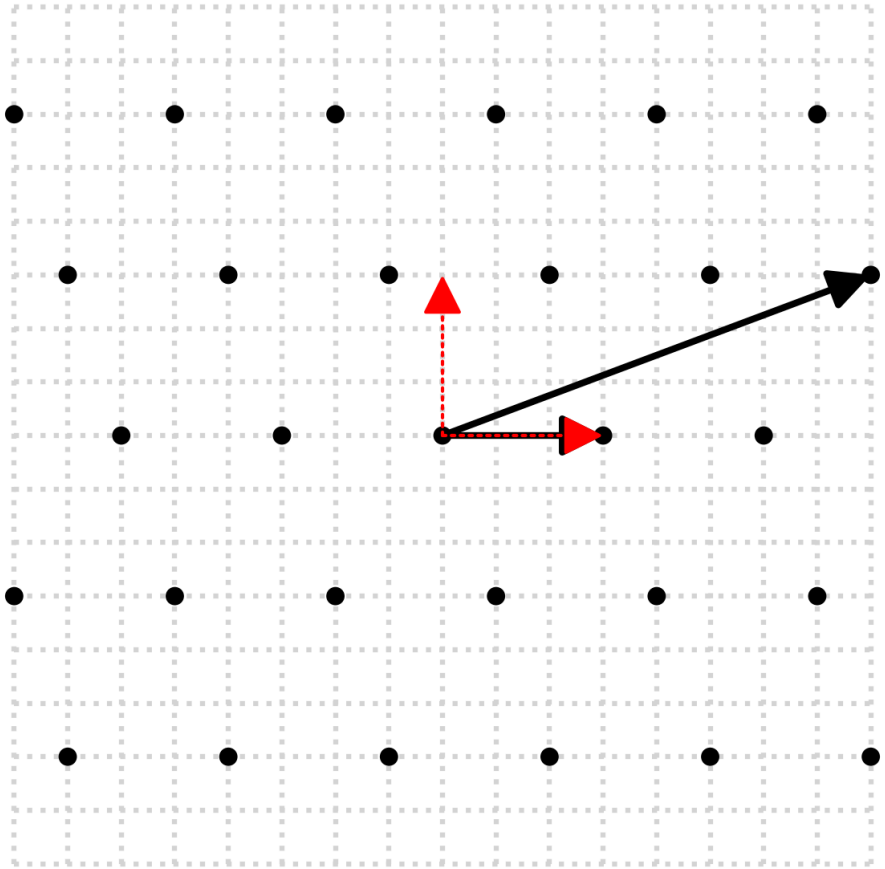

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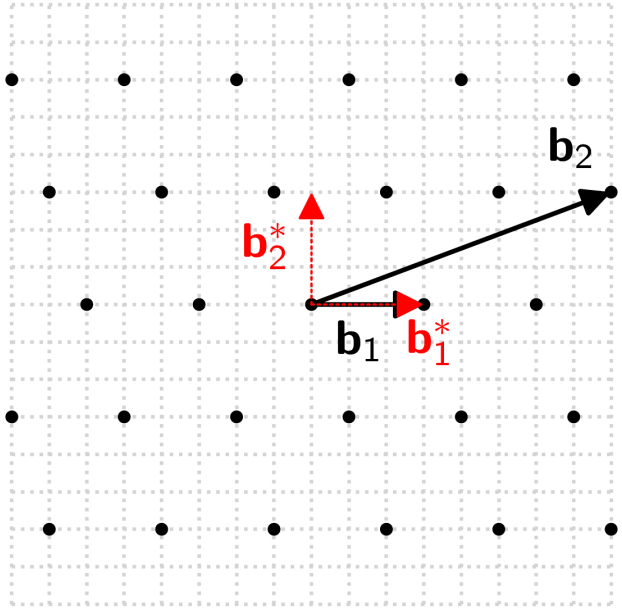

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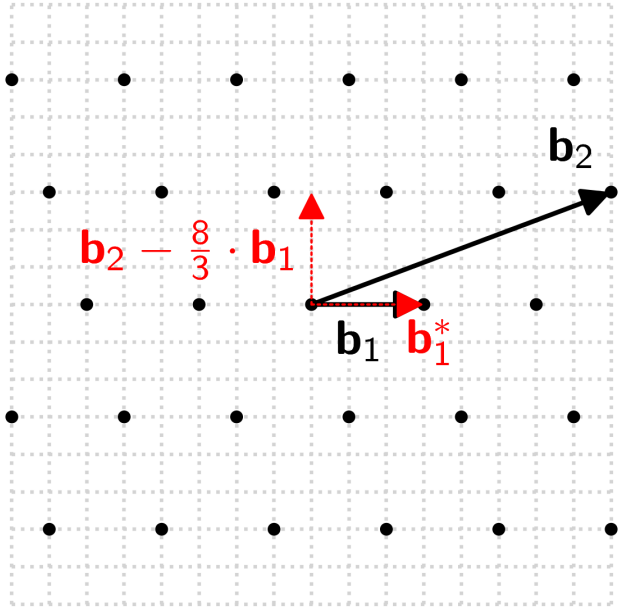

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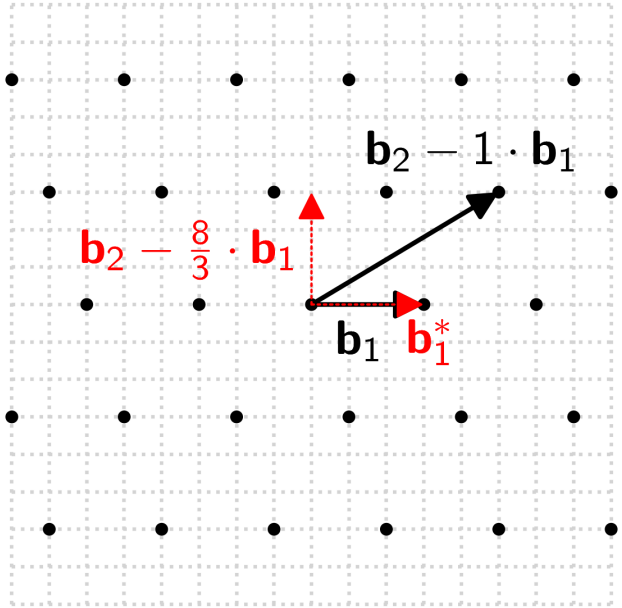

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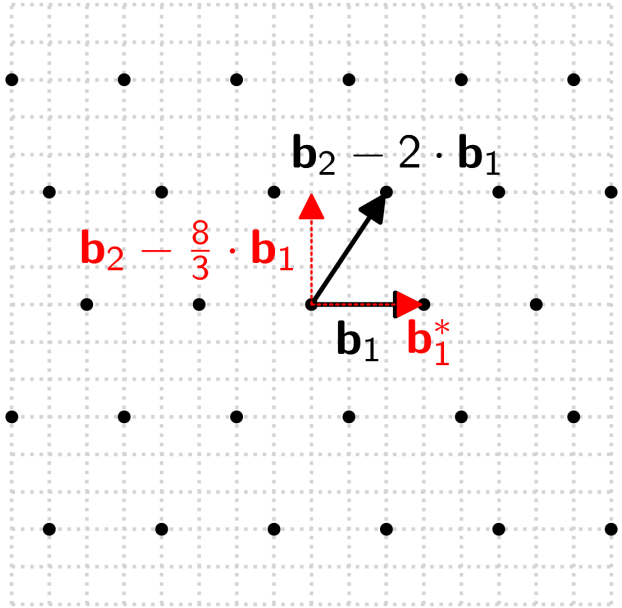

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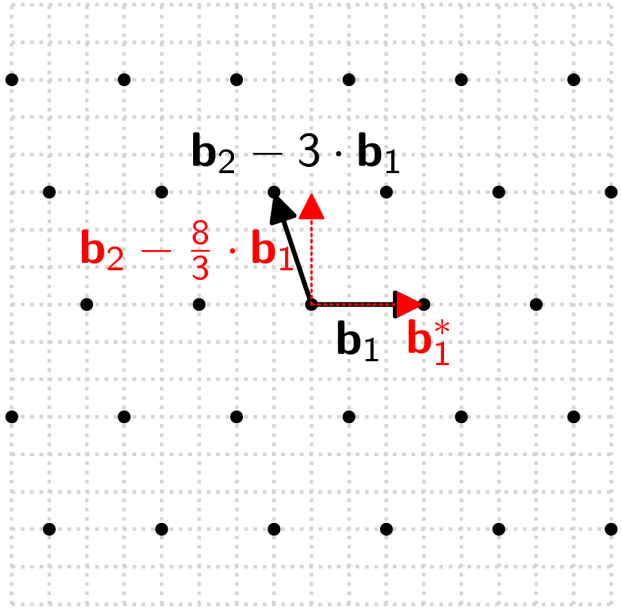

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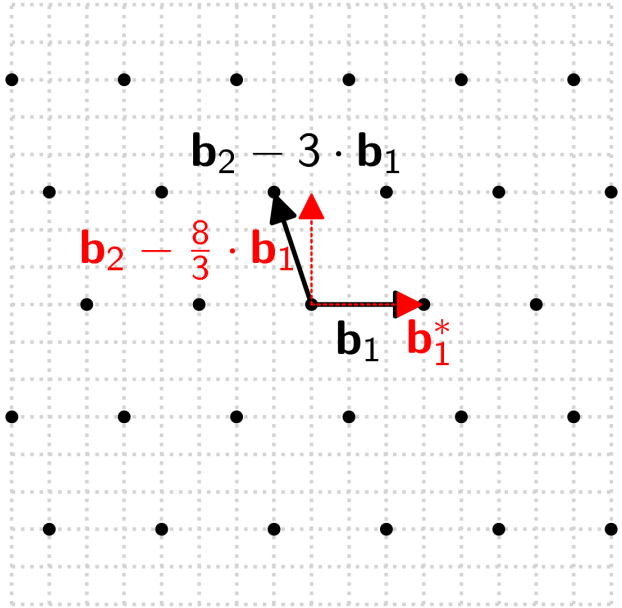

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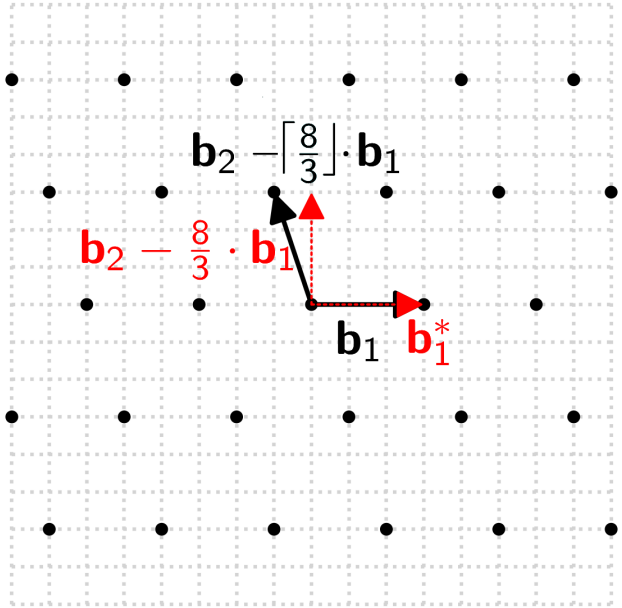

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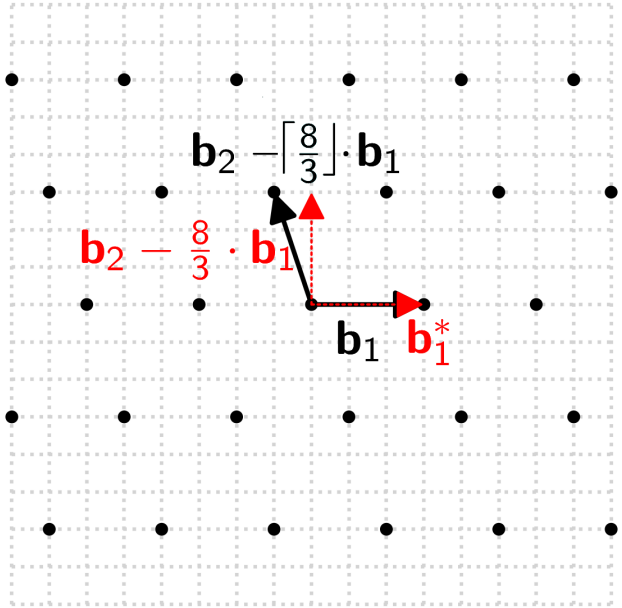

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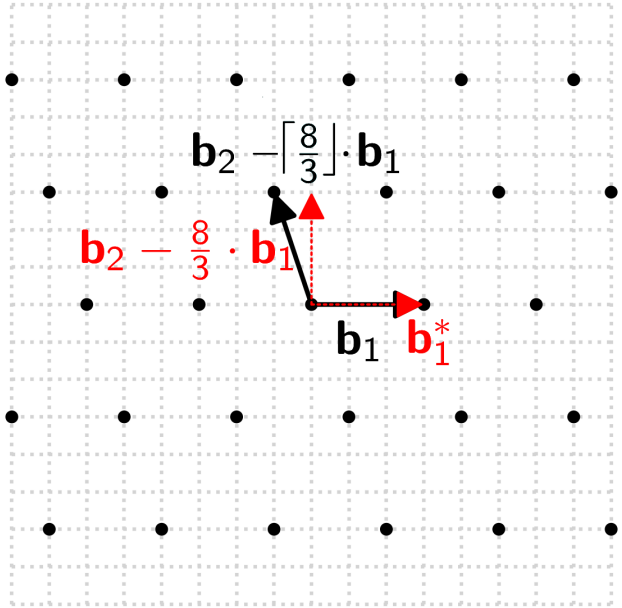

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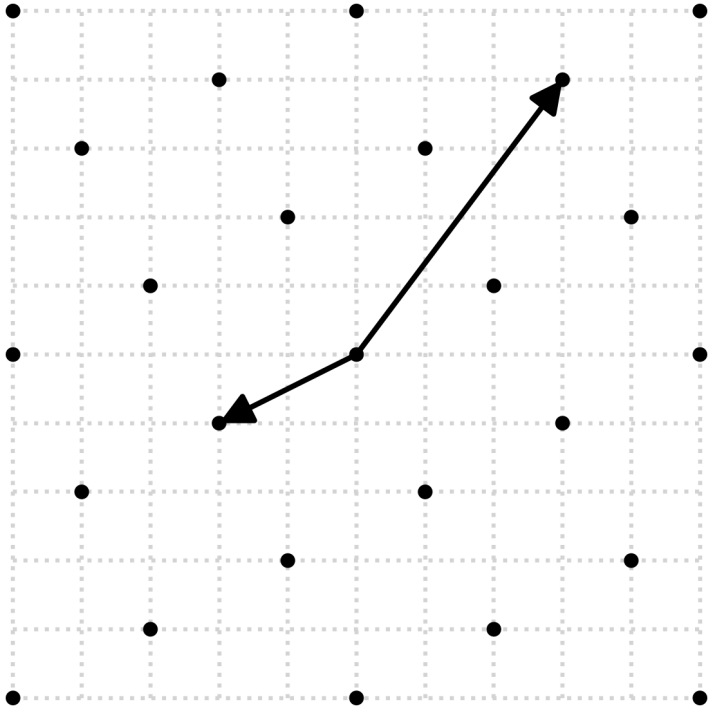
Definition: A basis is said to be **size-reduced** if:

$$\max_{1 \leq j < i \leq n} |\mu_{i,j}| \leq \frac{1}{2}$$

Why Size Reduction is Not Enough



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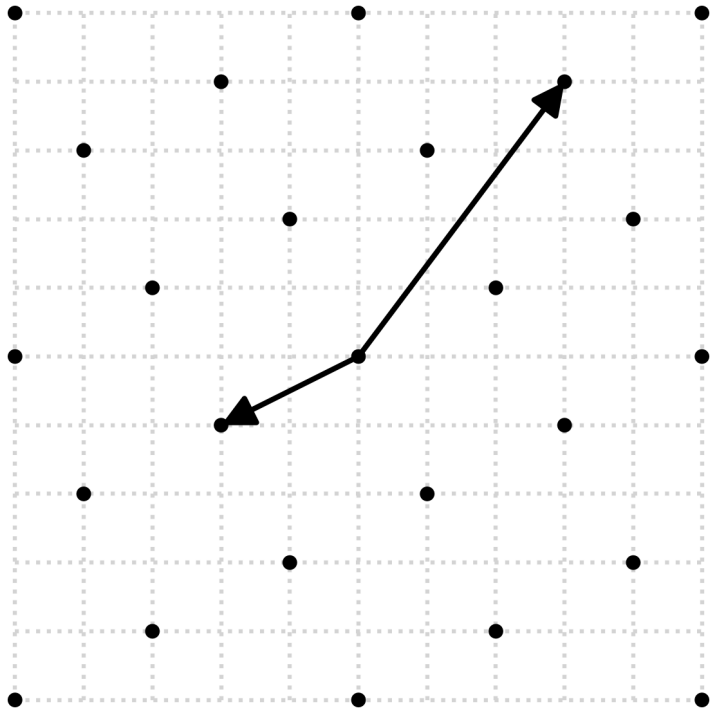

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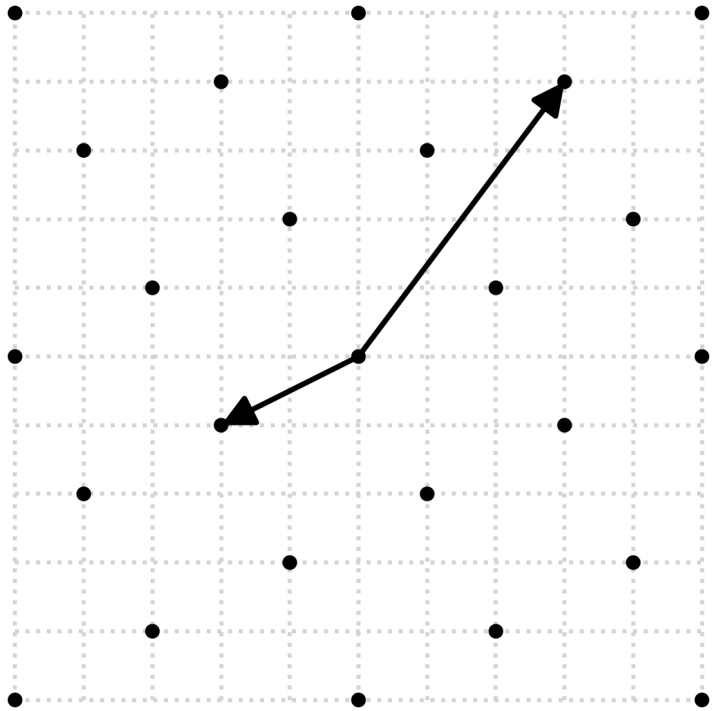

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*Length reduction alone **does not imply** almost-orthogonality!*

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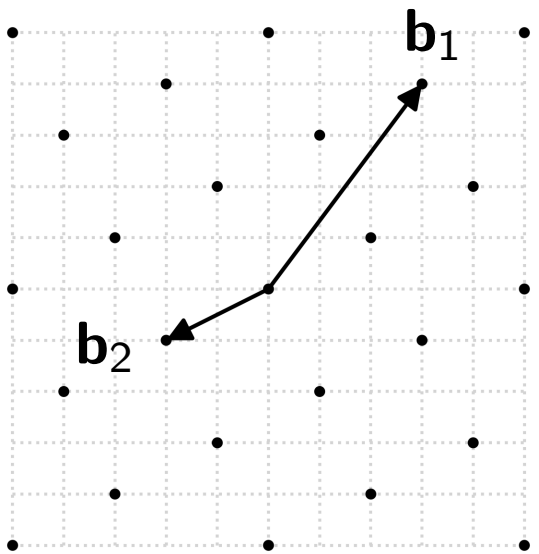

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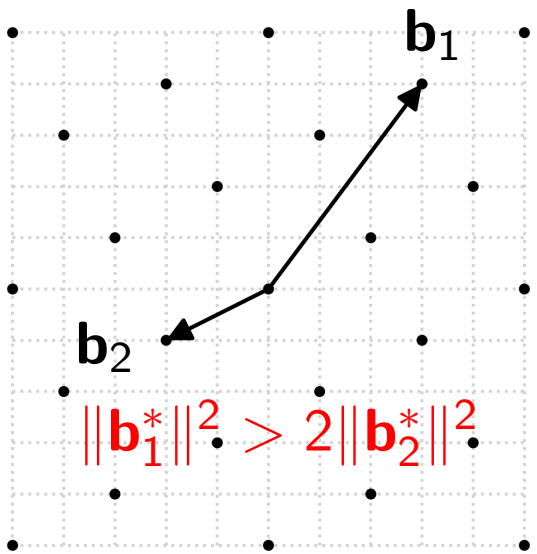

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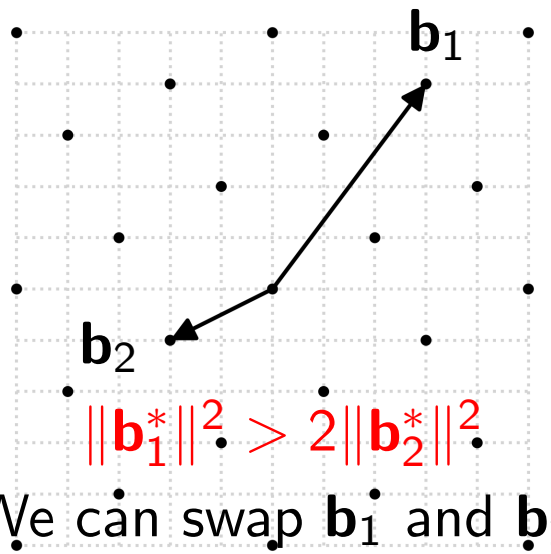

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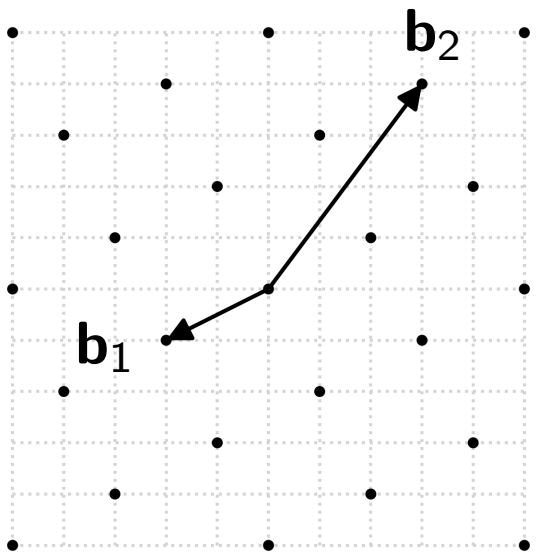

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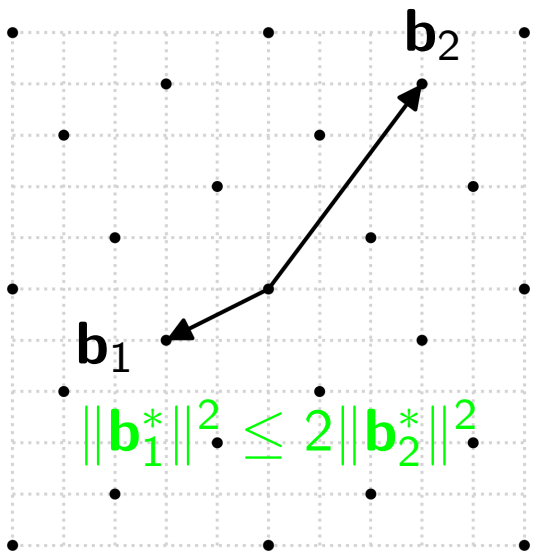

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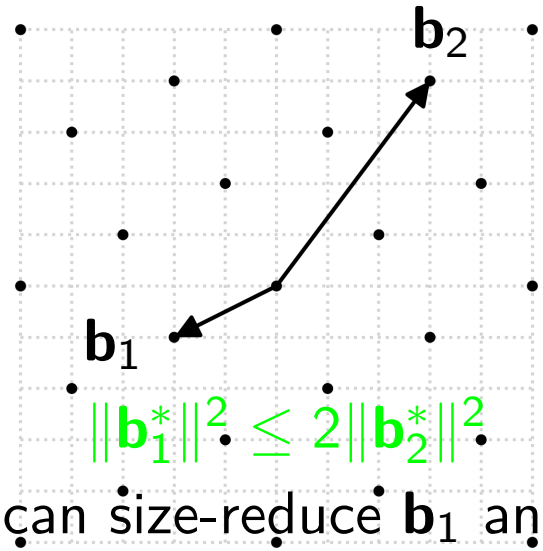

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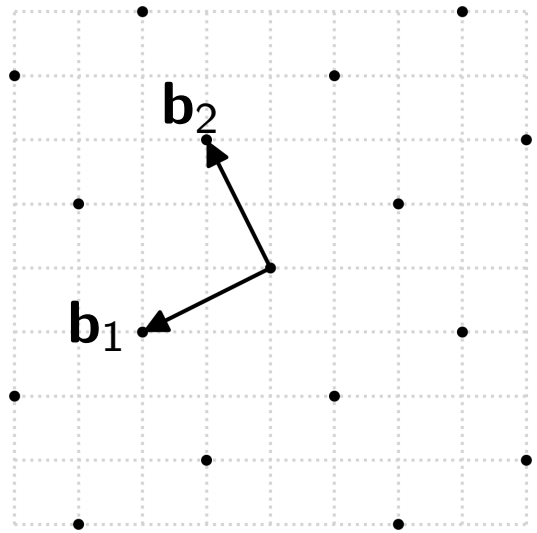

Definition: LLL– reduced Basis

A basis is called LLL–reduced if:

- It is size-reduced;
- It satisfies the Lovász condition.

Recap:The $\gamma - \mathrm{SVP}$ Problem

Definitions of λ_1, λ_2, ... are detailed in (Boudgoust 2023).

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Approximate Shortest Vector Problem ($\gamma - \mathrm{SVP}$)

Given a basis B of a lattice $\mathscr{L} \subset \mathbb{R}^n$ and an approximation factor $\gamma > 0$, **find a non-zero vector $\mathbf{v} \in \mathscr{L} \setminus \{\mathbf{0}\}$** such that:

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$\gamma = 1$	exact SVP — NP-hard
$\gamma = \mathrm{poly}(n)$	relevant for **lattice-based cryptography**
$\gamma = 2^{\mathcal{O}(n)}$	solvable in **polynomial time** via LLL

Lemma

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Proof. Let $(\mathbf{b}_i)_{1 \leq i \leq n}$ of the lattice $\mathscr{L}$, and write:

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Theorem. Let $(\mathbf{b}_i)_{1 \leq i \leq n}$ be a reduced basis of a lattice $\mathscr{L} \subseteq \mathbb{R}^n$, and let $\mathbf{b} \in \mathscr{L} \setminus \{0\}$. Then:

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$$\|\mathbf{b}_1\|^2 = \|\mathbf{b}_1^*\|^2 \leq 2\|\mathbf{b}_2^*\|^2 \leq 2^2\|\mathbf{b}_3^*\|^2 \leq \cdots \leq 2^{n-1}\|\mathbf{b}_n^*\|^2.$$

Thus,

$$\|\mathbf{b}\| \geq \min\{\|\mathbf{b}_1^*\|, \ldots, \|\mathbf{b}_n^*\|\} \geq 2^{-(n-1)/2}\|\mathbf{b}_1\|$$

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How can we compute a reduced basis in practice?

$\rightarrow$ Use the LLL (Lenstra 1982)(Lenstra, Lenstra, Lovasz) algorithm!

LLL Algorithm

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Input: A basis $B = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$
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$G \leftarrow copy(B)$
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while $i \leq n$ **do**
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Example

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Let's compute a LLL reduced basis of $\mathscr{L}(B)$ with

$$B := \begin{pmatrix} -2 & 2 & 1 \\ 3 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$$

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$$\overbrace{\begin{pmatrix} -2 & 2 & 1 \\ 3 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}}^{B} = \overbrace{\begin{pmatrix} 1 & 0 & 0 \\ -\frac{4}{9} & 1 & 0 \\ 0 & \frac{54}{101} & 1 \end{pmatrix}}^{U} \times \overbrace{\begin{pmatrix} -2 & 2 & 1 \\ \frac{19}{9} & \frac{8}{9} & \frac{22}{9} \\ \frac{88}{101} & \frac{154}{101} & -\frac{132}{101} \end{pmatrix}}^{B^*}$$

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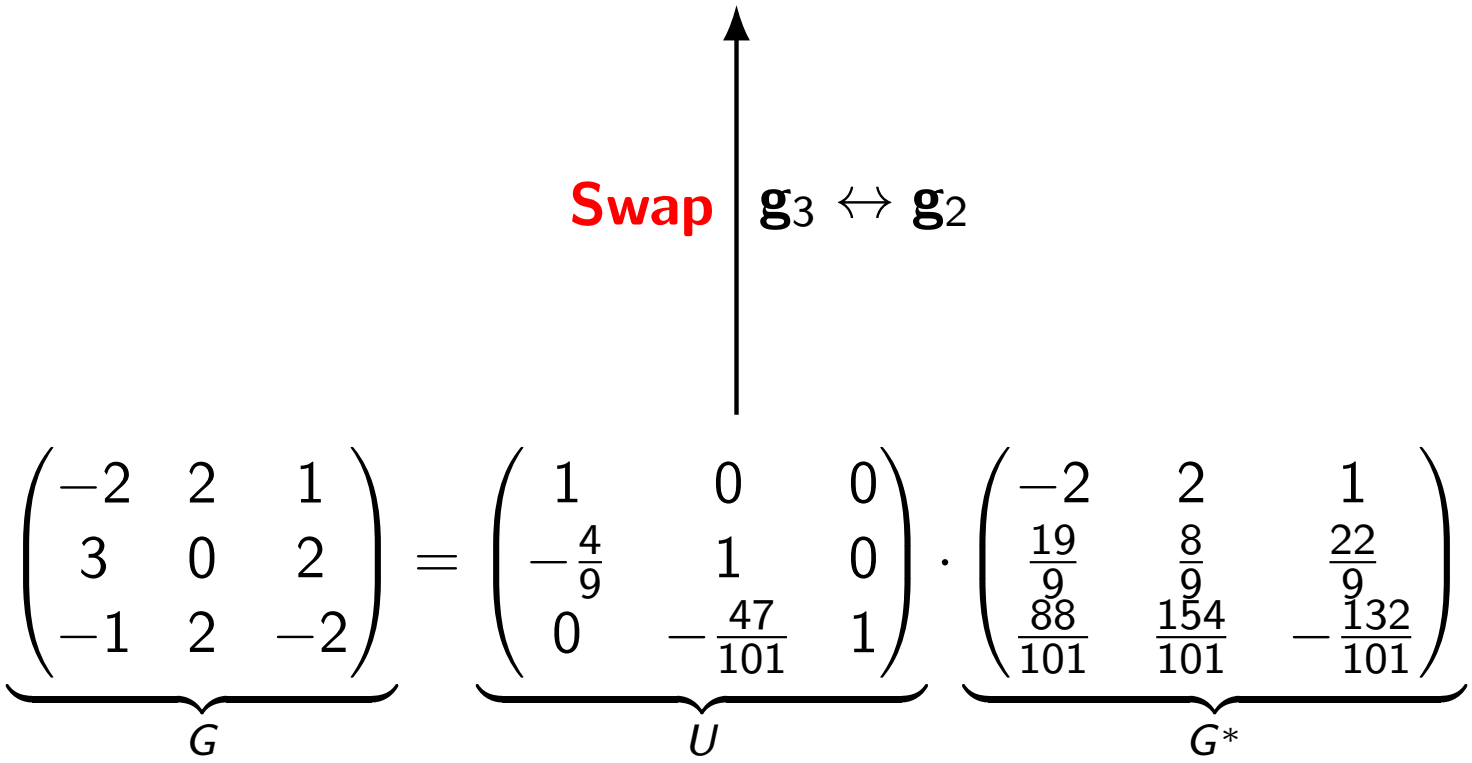
Swap
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The vector $(2, 2, 0)$ is a shortest nonzero vector in the lattice, hence:

$$\lambda_1(\mathscr{L}) = 2\sqrt{2}.$$

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Correctness

Key idea: Clearly, if the algorithm LLL **terminates**, the returned basis is by construction LLL-reduced.

Therefore, it remains ***to prove*** *that* LLL ***always terminates***.

How can we prove the termination of the algorithm?

$$\begin{pmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_{i-1} \\ \mathbf{b}_i \\ \vdots \\ \mathbf{b}_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ \mu_{2,1} & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \mu_{i-1,1} & \cdots & \mu_{i-1,i-2} & \ddots & \ddots & \vdots \\ \mu_{i,1} & \cdots & \mu_{i,i-2} & \mu_{i,i-1} & \ddots & \vdots \\ \vdots & & & & \ddots & 0 \\ \mu_{n,1} & \cdots & \cdots & \cdots & \mu_{n,n-1} & 1 \end{pmatrix} \times \begin{pmatrix} \mathbf{b}_1^* \\ \vdots \\ \mathbf{b}_{i-1}^* \\ \mathbf{b}_i^* \\ \vdots \\ \mathbf{b}_n^* \end{pmatrix}$$

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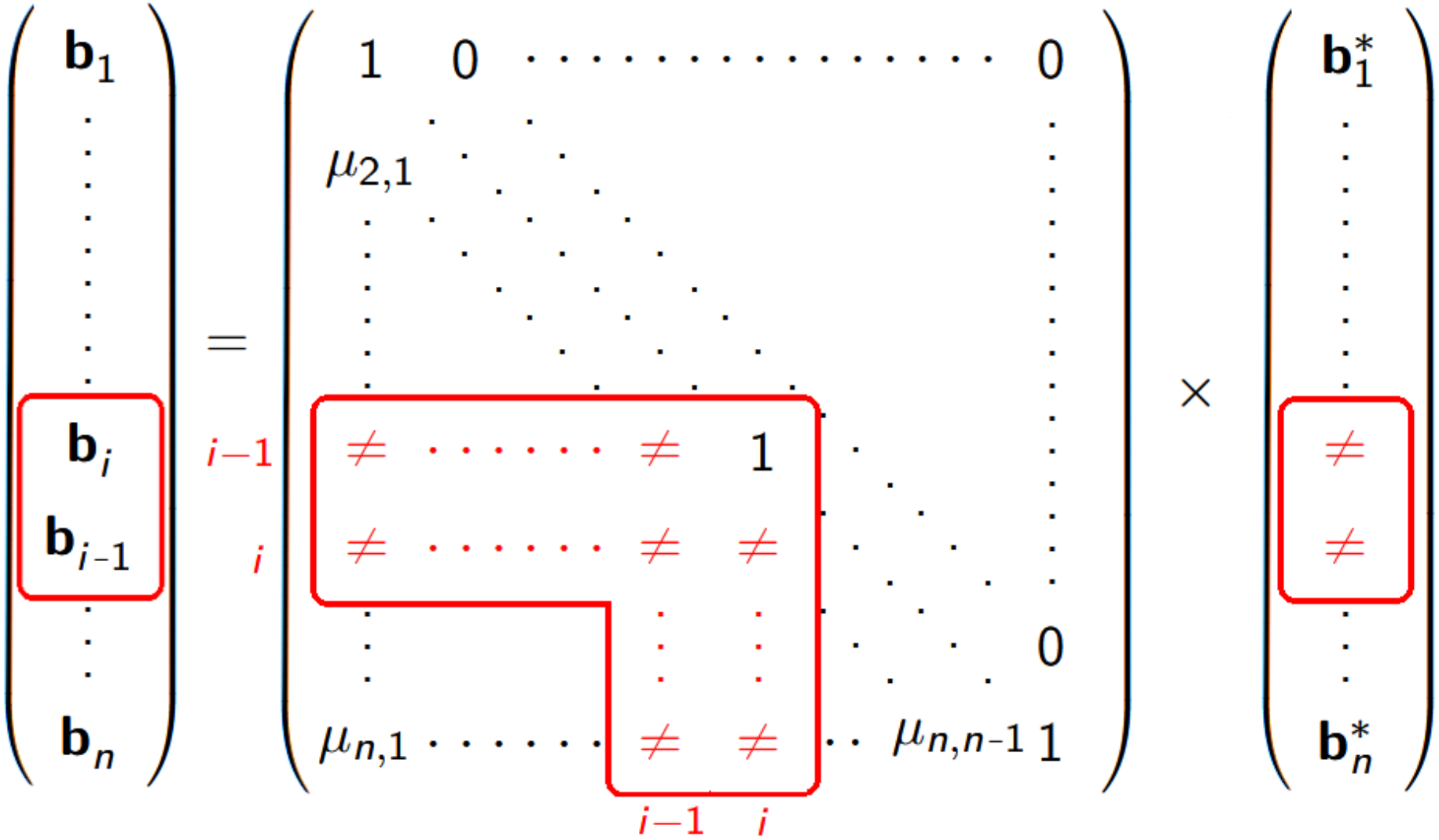

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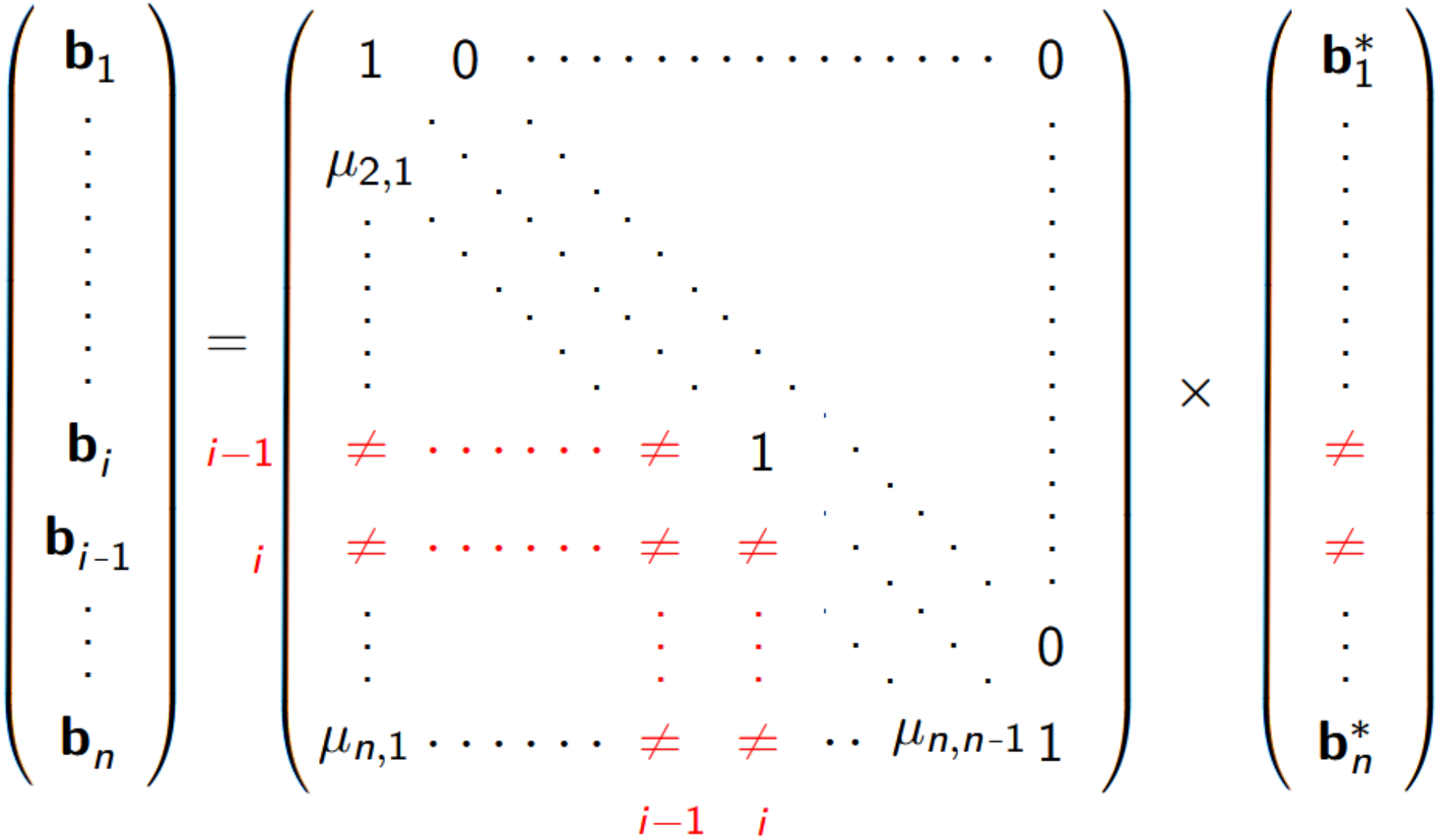

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$$\begin{pmatrix} \|\mathbf{b}_1\|^2 \\ \vdots \\ \|\mathbf{b}_i\|^2 \\ \|\mathbf{b}_{i-1}\|^2 \\ \vdots \\ \|\mathbf{b}_n\|^2 \end{pmatrix} = \begin{matrix} \\ \\ \\ i-1 \\ i \\ \\ \\ \end{matrix} \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \mu_{2,1}^2 & \ddots & \ddots & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\ \neq^2 & \cdots & \neq^2 & 1 & \ddots & \ddots & \vdots \\ \neq^2 & \cdots & \neq^2 & \neq^2 & \ddots & \ddots & \vdots \\ \vdots & & \vdots & \vdots & \ddots & \ddots & 0 \\ \mu_{n,1}^2 & \cdots & \neq^2 & \neq^2 & \cdots & \mu_{n,n-1}^2 & 1 \end{pmatrix} \times \begin{pmatrix} \|\mathbf{b}_1^*\|^2 \\ \vdots \\ \|\neq\|^2 \\ \|\neq\|^2 \\ \vdots \\ \|\mathbf{b}_n^*\|^2 \end{pmatrix}$$

$i-1 \quad i$

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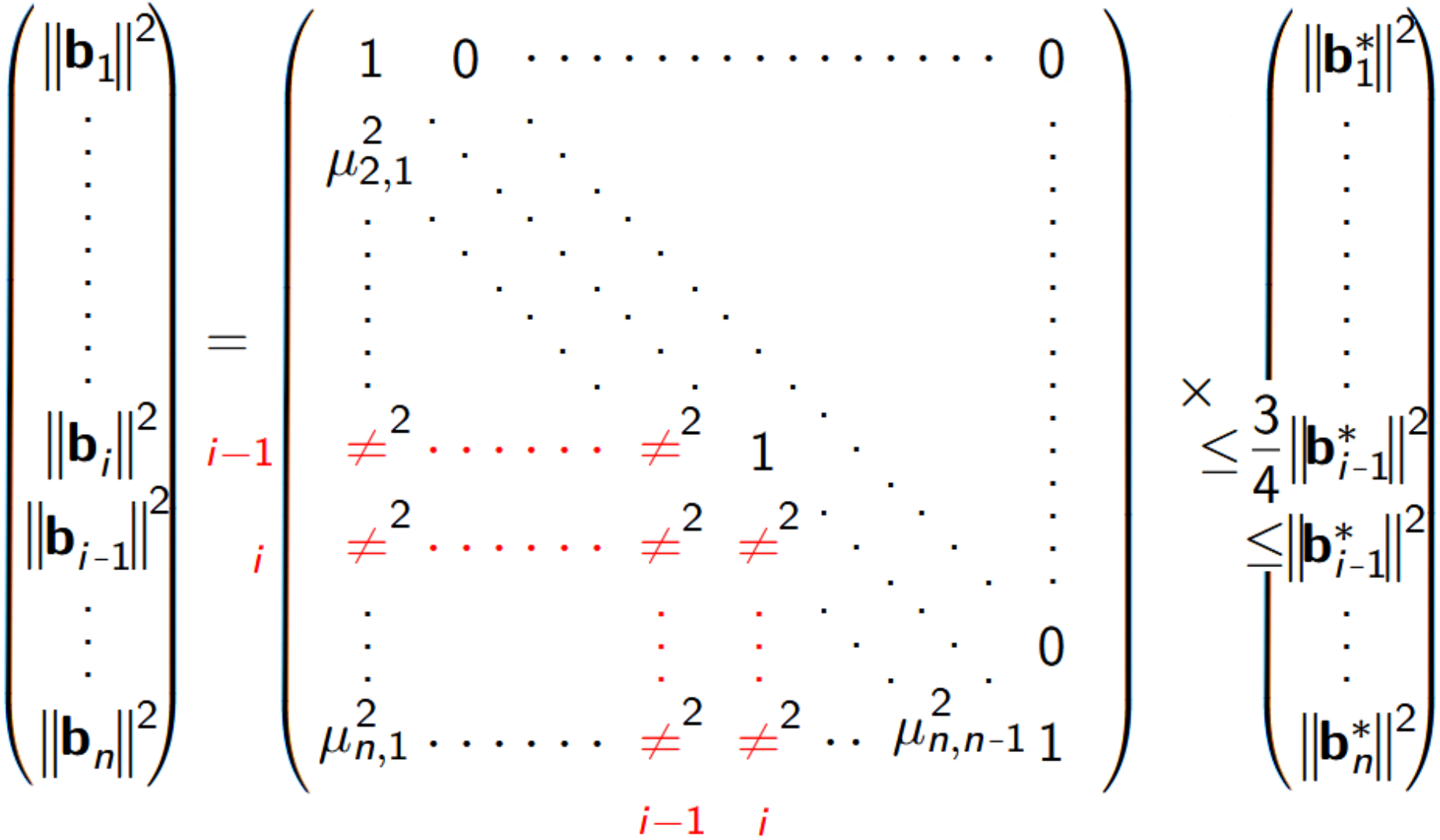

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If we **swap** $\mathbf{g}_i$ and $\mathbf{g}_{i-1}$:

$\|\mathbf{d}_{i-1}^*\|$ decrease by a $\frac{3}{4}$ factor, so $\mathbf{d_{i-1}}$ decrease by a $\frac{3}{4}$ factor.

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Termination proof

$$1 \leq \underbrace{\cdots}_{\mathcal{O}\left(\log\left(\max_{1 \leq i \leq n} \|\mathbf{g}_i\|\right)\right) \text{ steps}} \leq \frac{4}{3} D_1 \leq D_0 \leq \left(\max_{1 \leq i \leq n} \|\mathbf{g}_i\| \right)^{n(n-1)}$$

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3 **while** $i \leq n$ **do** $\mathcal{O}(n^2 log(A))$
4 **for** $j = i-1, i-2, \ldots, 1$ **do**
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- LLL uses $\mathcal{O}\left(n^2\right)$ arithmetic operations over rationals per iteration.

Theorem: Complexity of BasisReduction

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$\rightarrow$ LLL **compute** a reduced basis in **polynomial time**.

$\rightarrow$ LLL **solve** $2^{\mathcal{O}(n)} - \mathrm{SVP}$ in **polynomial time**.

Thank you for your attention!

Questions?

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