Towards Classsical Hardness of Module-LWE: The Linear Rank Case

Katharina Boudgoust Corentin Jeudy
Adeline Roux-Langlois Weiqiang Wen

Univ Rennes, CNRS, IRISA

Dromadaire, 9 November 2020, Online

Context of our contribution

The theoretical understanding

of the hardness assumptions that underlie

structured lattice-based cryptography.

Our main result [http://ia.cr/2020/1020]

A classical reduction

from a worst-case lattice problem

to the module learning with errors problem

with small modulus and linear rank.

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Outline

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Shortest Vector Problem (SVP) ...

A lattice Λ is a discrete additive subgroup of $\mathbb{R}^n$.
The **minimum** of Λ is $\lambda_1(\Lambda) = \min_{\mathbf{v} \in \Lambda \setminus \{\mathbf{0}\}} \|\mathbf{v}\|$.

Problem (Approximate Gap Shortest Vector Problem GapSVP_γ)

Let $\gamma \geq 1$. Given a lattice Λ and a parameter $\delta > 0$. Distinguish whether

$$\lambda_1(\Lambda) \leq \delta \quad \textit{or} \quad \lambda_1(\Lambda) > \gamma \cdot \delta.$$

If $\lambda_1(\Lambda) \in (\delta, \gamma \cdot \delta]$, any answer is correct.

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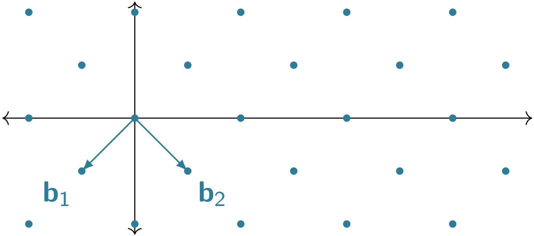

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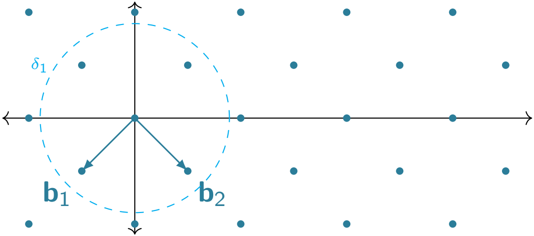

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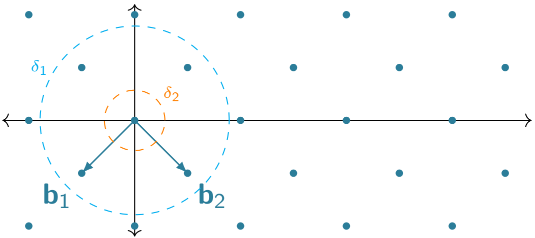

... In Module Lattices (Mod-GapSVP$_\gamma$)

Let K be a number field of degree n with R its ring of integers.

Think of K as $\mathbb{Q}[x]/(x^n+1)$ and of R as $\mathbb{Z}[x]/(x^n+1)$.

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An **ideal** I is a module of rank 1 and defines an **ideal lattice** $\sigma(I) \in \mathbb{R}^{1n}$.
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Concrete Example

Let K be the 4-th cyclotomic number field, having degree 2,
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Very low degree, **not** suited for real crypto schemes.

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Let $f = 3x+4$ and $g = -6x+1$ be elements in K.

- Addition: $f+g = -3x+5 \quad \in K$
- Multiplication:

$$\begin{aligned} f \cdot g &= (3x+4)(-6x+1) \\ &= -18x^2 + 3x - 24x + 4 \quad && (\text{use } x^2+1=0) \\ &= (3-24)x + (4+18) \\ &= -21x + 22 && \in K \end{aligned}$$

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Thus, $\sigma\left([(3x+4),(-6x+1)] \cdot \mathbb{Z}[x]/(x^2+1)\right)$ defines a **module lattice** of rank 2.

The Learning With Errors (LWE) Problem ...

Set $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$.

Given $\boxed{A} \sim U(\mathbb{Z}_q^{m \times d})$, $\boxed{b} \in \mathbb{Z}_q^m$, $\boxed{s} \sim U(\mathbb{Z}_q^d)$ and $\boxed{e} \sim D_{\mathbb{Z}^m,\alpha}$ s.t.

$$m\left\{\underbrace{A}_{d}\right. , \quad A \, s + e = b$$

Search: Find secret $\boxed{s}$.

Decision: Distinguish from ($\boxed{A}$, $\boxed{b}$), where $\boxed{b} \sim U(\mathbb{Z}_q^m)$.

... With Structure (Module-LWE)

Replace $\mathbb{Z}$ by R, the ring of integers of some number field K of degree n.
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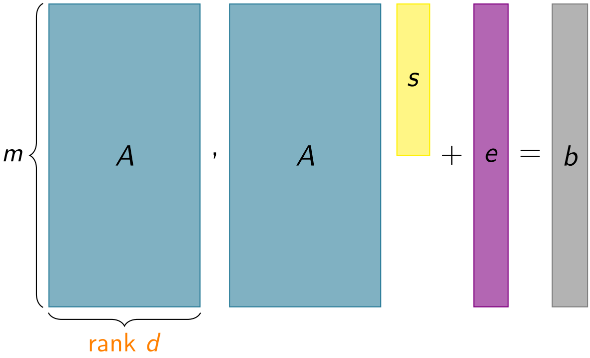

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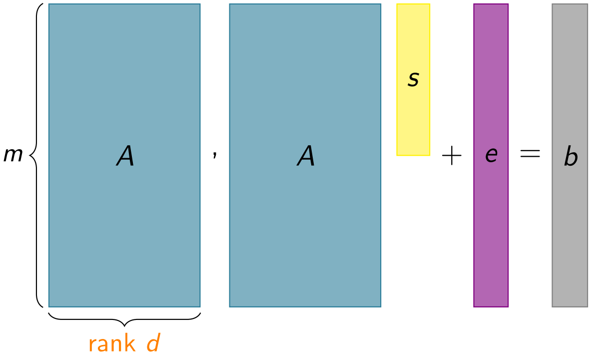

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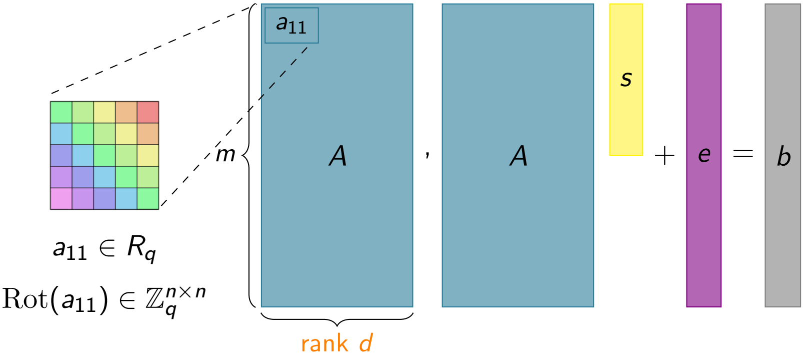

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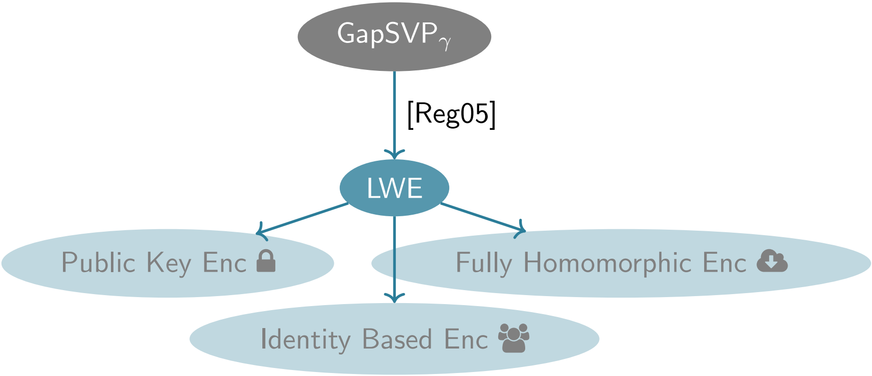
Overview

Motivation: What we know for LWE



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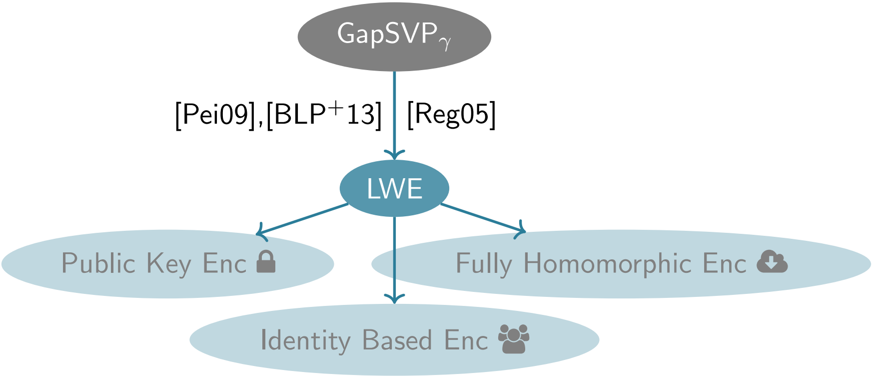
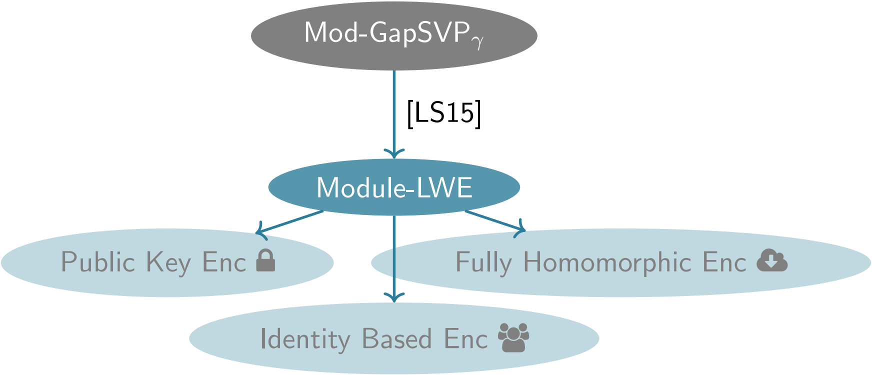


- [Reg05]: quantum reduction, LWE modulus q is poly-large
- [Pei09]: classical reduction, LWE modulus q is exp-large
- [BLP$^+$13]: classical reduction **and** LWE modulus q is poly-large

Motivation: And what we know for Module-LWE



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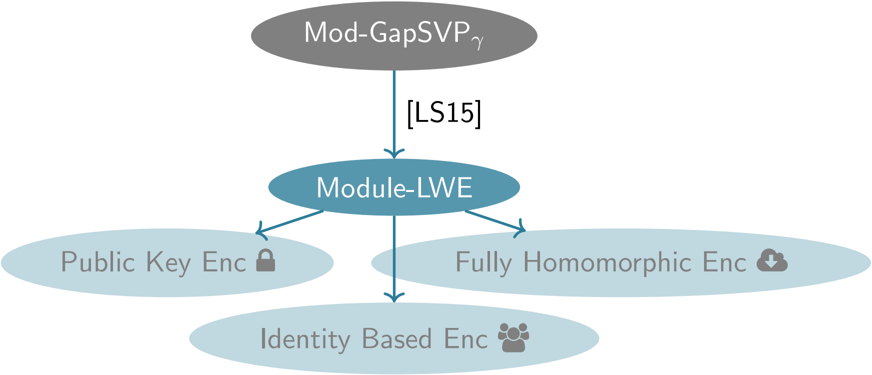

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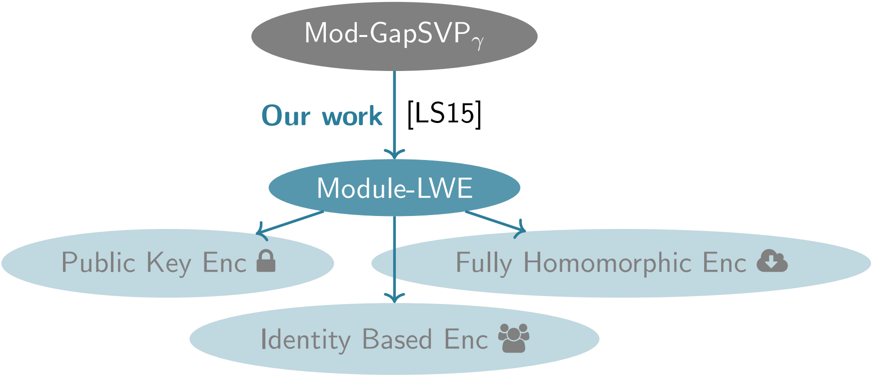

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- **Our work**: classical **and** modulus is poly-large **and** decisional, **but** rank linear

Why do we care?

Multiple third-round candidates for the NIST standardization process are based on Module-LWE (and variants)

Public Key Encryption

- Crystals-Kyber: Module-LWE
- Saber: Module-LWR (deterministic variant)

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However, they only require very small ranks, between 2 and 5, much smaller than n.

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High level idea following [BLP$^+$13]

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  Trivial decision-to-search reduction, intelligent noise flooding applied to LWE-analogue [GKPV10], much simpler than [BLP$^+$13].

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Today: We will only see Step 2.

From the idea to the full proof ...

$$
\begin{tikzcd}
\text{Mod-GapSVP}_{\gamma} \arrow[d, "\text{Thm. } \boxed{4}", "\ell\geq 2, q\geq 2^{\frac{n\ell}{2}}, \gamma\alpha\geq n\ell\sqrt{\ell}"'] & \\
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& \text{bin-M-SLWE}^{m}_{n,d,p,D_{\beta'}} \arrow[d] \\
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The secret $\mathbf{s} \in R_2^d$ is binary and the secret $\mathbf{s}' \in R_q^{\ell}$ is modulo q.

$$m\left\{\underbrace{\mathbf{A}}_{d}\right. , \mathbf{A}\,\mathbf{s} + \mathbf{e}$$

Module-LWE with binary secret

Tikz-Credits to Corentin

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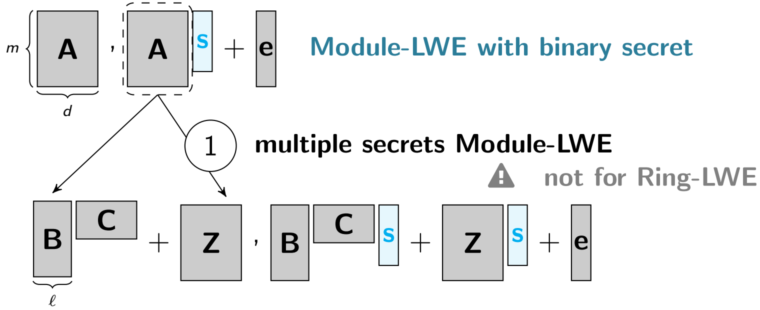

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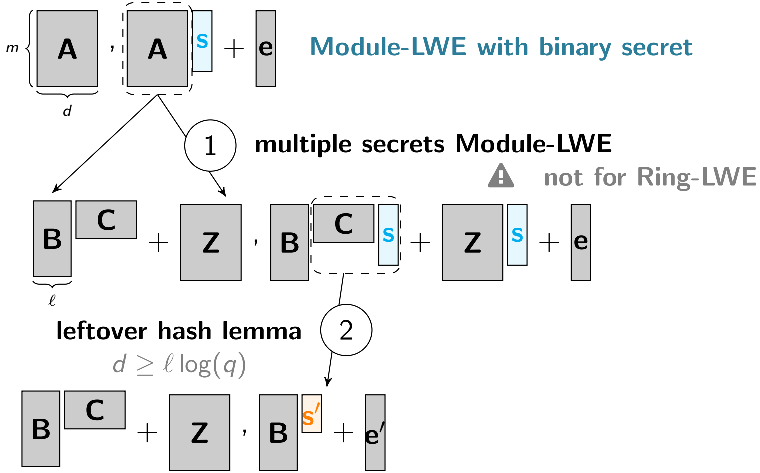

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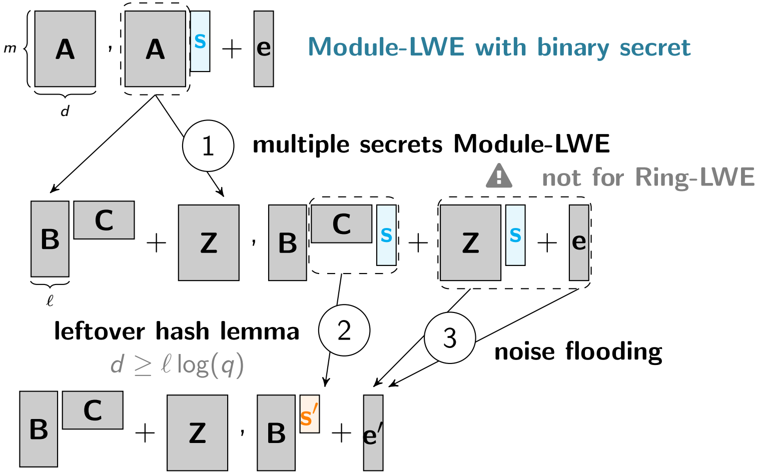

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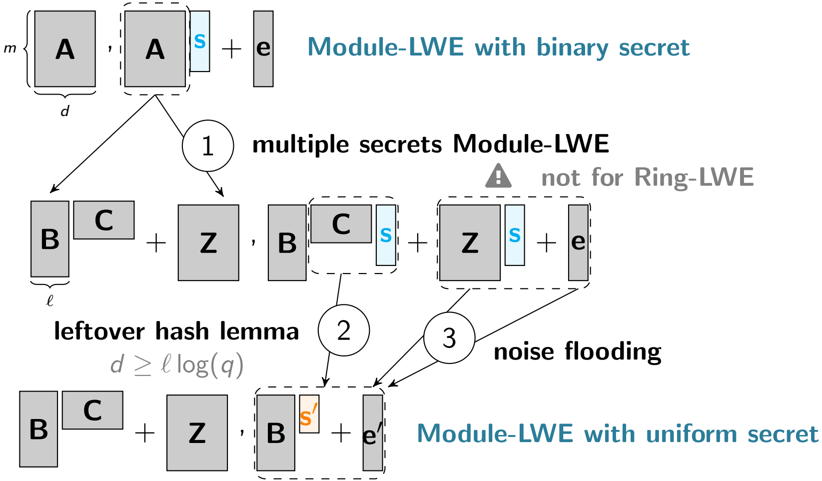

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Improved noise flooding using Rényi Divergence 1/2

Let P, Q be discrete probability distributions.

In [GKPV10]: Statistical Distance

$$SD(P,Q) = \frac{1}{2} \sum_{x \in \mathrm{Supp}(P)} |P(x) - Q(x)|$$

In our work: Rényi Divergence

$$RD(P,Q) = \sum_{x \in \mathrm{Supp}(P)} \frac{P(x)^2}{Q(x)}$$

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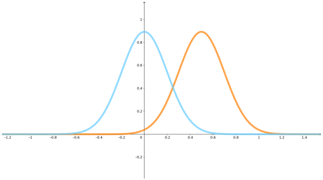

Example: two Gaussians D_β and $D_{\beta,s}$,

$$RD(D_\beta, D_{\beta,s}) = \exp\left(\frac{2\pi\|s\|^2}{\beta^2}\right)$$

$$SD(D_\beta, D_{\beta,s}) = \frac{\sqrt{2\pi}\|s\|}{\beta}$$

Improved noise flooding using Rényi Divergence 2/2

Both fulfill the **probability preservation property** for an event E:

[GKPV10]:	$P(E)$	$\leq$	$SD(P,Q) + Q(E)$	(additive)
Our work:	$P(E)^2$	$\leq$	$RD(P,Q) \cdot Q(E)$	(multiplicative)

We need: $Q(E)$ negligible $\Rightarrow P(E)$ negligible

Thus: $SD(P,Q) \stackrel{!}{=}$ negligible and $RD(P,Q) \stackrel{!}{=}$ **constant**

Improved noise flooding using Rényi Divergence 2/2

Both fulfill the **probability preservation property** for an event E:

$$\begin{array}{llcll} \text{[GKPV10]:} & P(E) & \leq & SD(P,Q) + Q(E) & \text{(additive)} \\ \textbf{Our work}: & P(E)^2 & \leq & RD(P,Q) \cdot Q(E) & \text{(multiplicative)} \end{array}$$

We need: $Q(E)$ negligible $\Rightarrow$ $P(E)$ negligible

Thus: $SD(P,Q) =^! \text{negligible}$ and $RD(P,Q) =^!$ **constant**

Back to example: two Gaussians D_β and $D_{\beta,\mathbf{s}}$ with $\|\mathbf{s}\| \leq \alpha$

$$\begin{array}{lll} SD(D_\beta, D_{\beta,\mathbf{s}}) & = \frac{\sqrt{2\pi}\|\mathbf{s}\|}{\beta} & \Rightarrow \alpha/\beta \leq \text{negligible} \\ RD(D_\beta, D_{\beta,\mathbf{s}}) & = \exp\left(\frac{2\pi\|\mathbf{s}\|^2}{\beta^2}\right) \approx 1 + \frac{2\pi\|\mathbf{s}\|^2}{\beta^2} & \Rightarrow \alpha/\beta \leq \textbf{constant} \end{array}$$

(Taylor expansion at 0)

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Back to example: two Gaussians D_β and $D_{\beta,s}$ with $\|s\| \leq \alpha$

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⚠ Rényi Divergence only for search problems.

High level idea following [BLP+13]

- Step 1: Classical reduction from Mod-GapSVP$_\gamma$ to decisional Module-LWE with exp-large modulus
  Adapting and merging module variants of [Pei09] (classical) and [PRS17] (decisional), using the Oracle Hidden Center Problem.
- Step 2: Reduction from decisional Module-LWE **and** search Module-LWE to search Module-LWE with **binary secret**
  Trivial decision-to-search reduction, intelligent noise flooding applied to LWE-analogue [GKPV10], much simpler than [BLP+13].
- Step 3: Modulus reduction from exp-large to poly-large modulus for Module-LWE with **binary secret**
  Using [AD17], computing bounds on singular values of rotation matrix, loss in the reduction depends on the norm of the secret.

Today: We only saw Step 2.

Overview

Further work and open questions

Related work

- Other small secret distributions (HNF, Entropic LWE)

Work in progress

- Refined proof for hardness of binary Module-LWE
  Independent of number of samples

Open questions ?

- Smaller rank, in particular rank equals 1 (Ring-LWE)
- Other number fields than power-of-two cyclotomics (bounds on singular values on the rotation matrix)

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Thank you.

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Backup

Concrete Example Continued

Let K be the 4-th cyclotomic number field, having degree 2, $K = \mathbb{Q}[x]/(x^2+1)$, where $x^2+1 = (x-i)(x+i)$.

Let $f = 3x+4$ and $g = -6x+1$ be elements in K.

The **canonical** embedding σ is given by
$\sigma(f) = (3i+4, -3i+4) \in \mathbb{C}^2$ and $\sigma(g) = (-6i+1, 6i+1) \in \mathbb{C}^2$.
Multiplication is component-wise (fast), thanks to the symmetries the image $\sigma(f)$ can be represented by a 2-dim real vector $\sigma_{\mathbb{R}}(f) \in \mathbb{R}^2$.

The **coefficient** embedding τ given by
$\tau(f) = (4,3)$ and $\tau(g) = (1,-6)$.
Multiplication via convolution product (slow)

Relation between σ and τ via the **Vandermonde matrix**:

$$\sigma(f) = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \tau(f).$$

Used to speed up computations in Module-LWE.