Towards Classsical Hardness of Module-LWE: The Linear Rank Case

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Our main result [http://ia.cr/2020/1020]

A classical reduction

from a worst-case lattice problem

to the module learning with errors problem

with small modulus and linear rank.

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GapSVP$_{\gamma(n)}$ in module lattices → worst-case lattice problem

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rank $\geq 2n$

Outline

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Shortest Vector Problem (SVP) ...

For a lattice $\Lambda \subset \mathbb{R}^n$ set $\lambda_1(\Lambda) = \min_{\mathbf{v} \in \Lambda \setminus \{\mathbf{0}\}} \|\mathbf{v}\|$.

Problem (Approximate Gap Shortest Vector Problem GapSVP_γ)

Let $\gamma \geq 1$. Given a lattice Λ and a parameter $\delta > 0$. Distinguish whether

$$\lambda_1(\Lambda) \leq \delta \quad \textit{or} \quad \lambda_1(\Lambda) > \gamma \cdot \delta.$$

If $\lambda_1(\Lambda) \in (\delta, \gamma \cdot \delta]$, any answer is correct.

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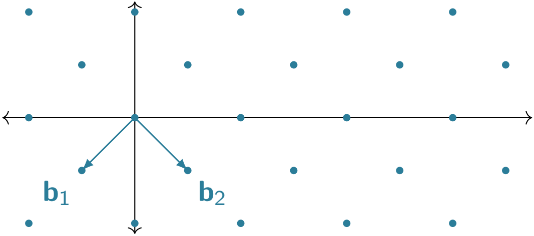

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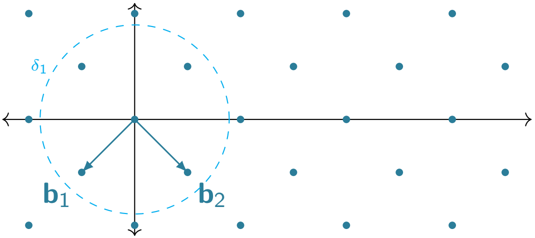

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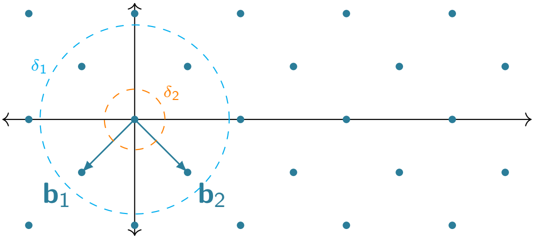

... In Module Lattices (Mod-GapSVP$_\gamma$)

Let K be a number field of degree n with R its ring of integers.
Think of K as $\mathbb{Q}[x]/(x^n + 1)$ and of R as $\mathbb{Z}[x]/(x^n + 1)$.

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An R-**module** M of rank d defines via σ a **module lattice** $\sigma(M) \in \mathbb{R}^{dn}$.
An **ideal** I is a module of rank 1 and defines an **ideal lattice** $\sigma(I) \in \mathbb{R}^{1n}$.
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Problem (Mod-GapSVP$_\gamma$)

Let $\gamma \geq 1$. Given a **module lattice** $\Lambda = \sigma(M)$ *and a parameter $\delta > 0$. Distinguish whether*

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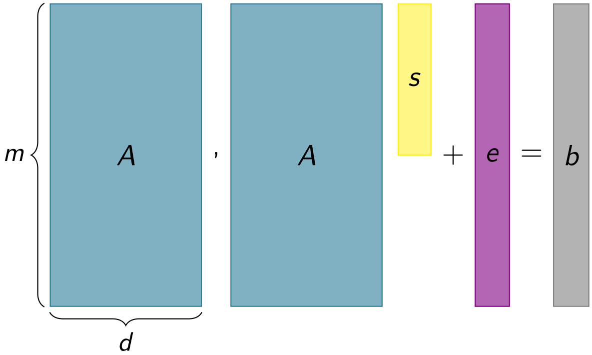
The Learning With Errors (LWE) Problem ...

Set $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$.

Given $A \sim U(\mathbb{Z}_q^{m \times d})$, $b \in \mathbb{Z}_q^m$, $s \sim U(\mathbb{Z}_q^d)$ and $e \sim D_{\mathbb{Z}^m,\alpha}$ s.t.



Search: Find secret s.

Decision: Distinguish from (A, b), where $b \sim U(\mathbb{Z}_q^m)$.

... With Structure (Module-LWE)

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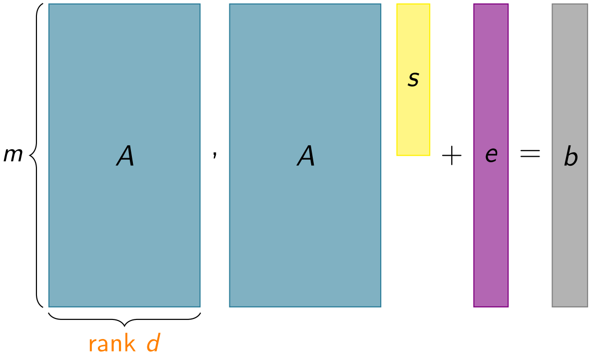

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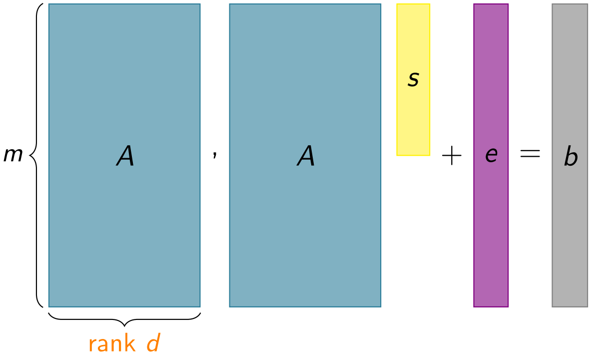

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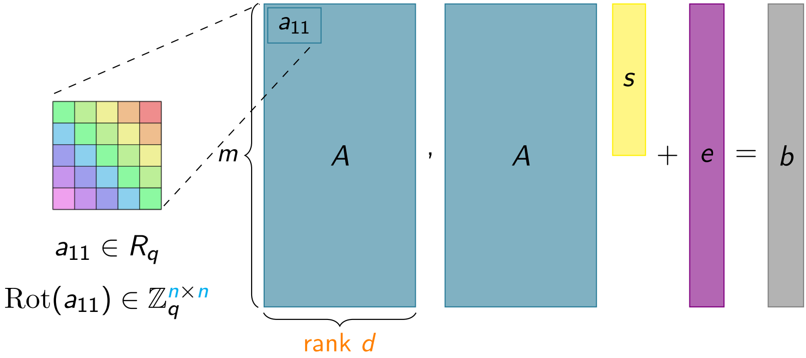

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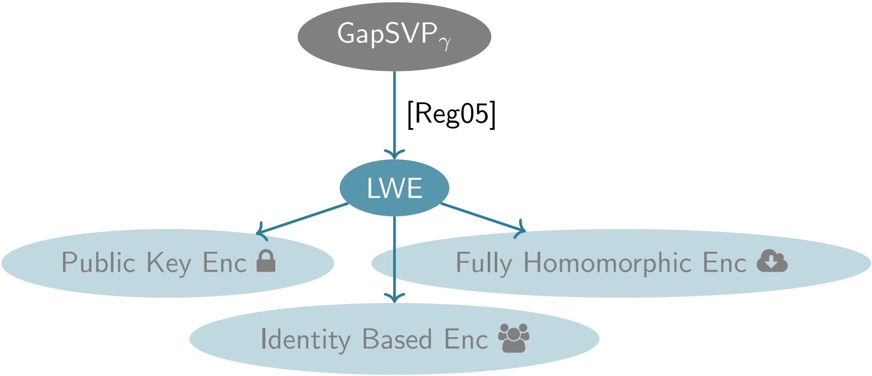
Overview

Motivation: What we know for LWE



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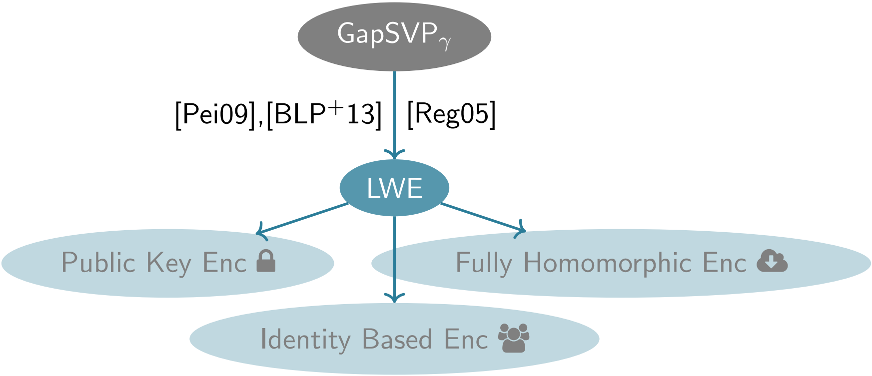
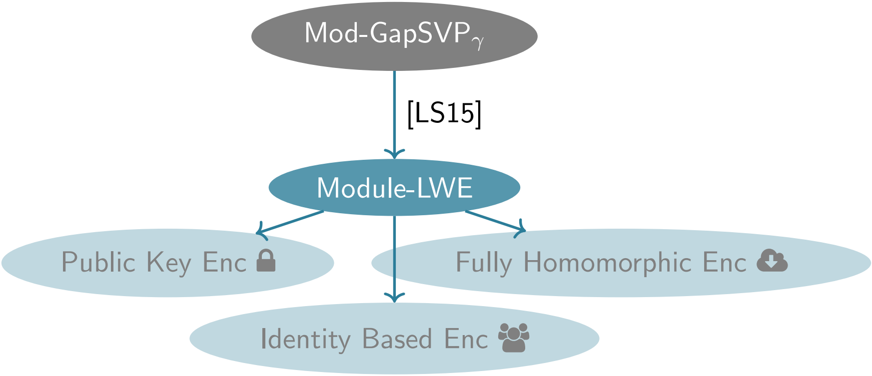


- [Reg05]: quantum reduction, LWE modulus q is poly-large
- [Pei09]: classical reduction, LWE modulus q is exp-large
- [BLP$^+$13]: classical reduction **and** LWE modulus q is poly-large

Motivation: And what we know for Module-LWE



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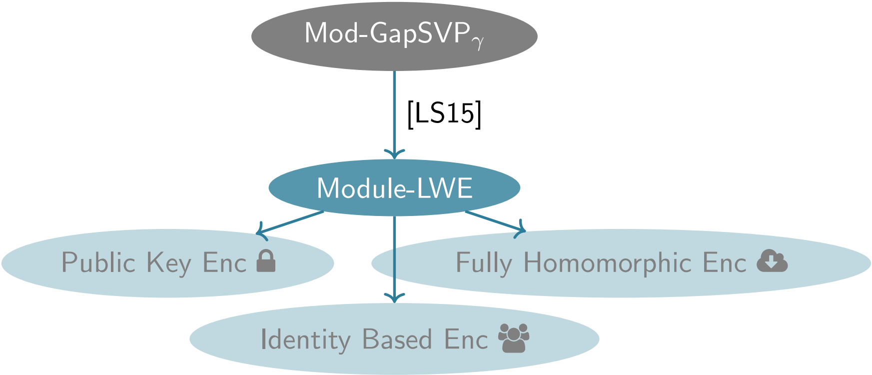

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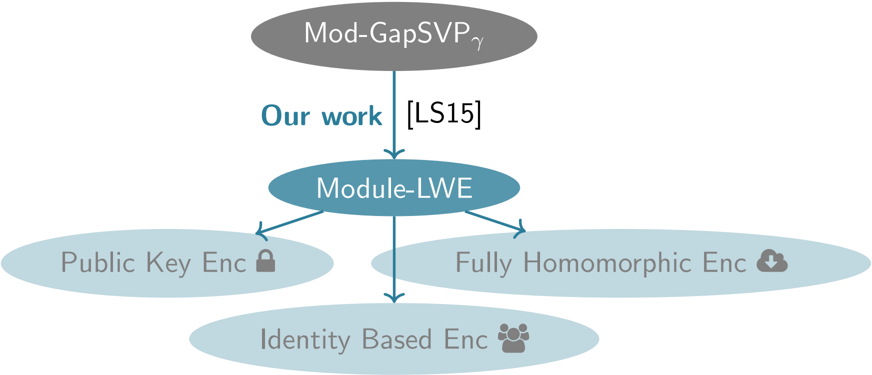

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- **Our work**: classical **and** modulus is poly-large **and** decisional, **but** rank linear

Why do we care?

Multiple third-round candidates for the NIST standardization process are based on Module-LWE (and variants)

Public Key Encryption

- Crystals-Kyber: Module-LWE
- Saber: Module-LWR (deterministic variant)

Digital Signature

- Crystals-Dilithium: Module-LWE

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However, they only require very small ranks, between 2 and 5, much smaller than n.

Overview

High level idea following [BLP+13]

- Step 1: Classical reduction from Mod-GapSVP$_\gamma$ to decisional Module-LWE with exp-large modulus

  Adapting and merging module variants of [Pei09] (classical) and [PRS17] (decisional), using the Oracle Hidden Center Problem.

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  Trivial decision-to-search reduction, intelligent noise flooding using the **Rényi Divergence** applied to LWE-analogue [GKPV10], much simpler than [BLP+13].

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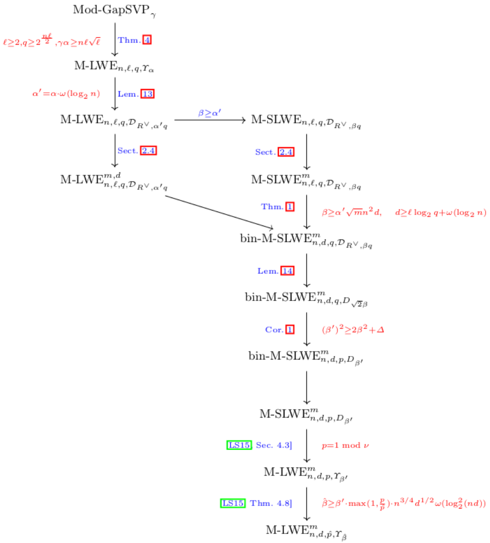
- Step 3: Modulus reduction from exp-large to poly-large modulus for Module-LWE with **binary secret**
  Using [AD17], computing bounds on singular values of rotation matrix, loss in the reduction depends on the norm of the secret, optimal for **binary**.

From the idea to the full proof ...



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Further work and open questions

Work in progress

- Refined proof for hardness of binary Module-LWE
  Independent of number of samples

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- Smaller rank, in particular rank equals 1 (Ring-LWE)
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Thank you.

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Backup

Concrete Example 1/2

Let K be the 4-th cyclotomic number field, having degree 2, $K = \mathbb{Q}[x]/(x^2+1)$, where $x^2+1 = (x-i)(x+i)$.

Very low degree, **not** suited for real crypto schemes.

Let $f = 3x+4$ and $g = -6x+1$ be elements in K.

- Addition: $f + g = -3x + 5 \in K$
- Multiplication:

$$
\begin{aligned}
f \cdot g &= (3x+4)(-6x+1) \\
&= -18x^2 + 3x - 24x + 4 \quad (\text{use } x^2+1=0) \\
&= (3-24)x + (4+18) \\
&= -21x + 22 \quad \in K
\end{aligned}
$$

Then, for every $f \in K$, the canonical embedding σ is given by $\sigma(f) = (f(i), f(-i)) \in \mathbb{C}^2$.

For example $\sigma(3x+4) = (3i+4, -3i+4)$.

Thus, $\sigma\left([(3x+4),(-6x+1)] \cdot \mathbb{Z}[x]/(x^2+1)\right)$ defines a **module lattice** of rank 2.

Concrete Example 2/2

Let K be the 4-th cyclotomic number field, having degree 2,
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Let $f = 3x + 4$ and $g = -6x + 1$ be elements in K.

The **canonical** embedding σ is given by
$\sigma(f) = (3i+4, -3i+4) \in \mathbb{C}^2$ and $\sigma(g) = (-6i+1, 6i+1) \in \mathbb{C}^2$.
Multiplication is component-wise (fast), thanks to the symmetries the image $\sigma(f)$ can be represented by a 2-dim real vector $\sigma_{\mathbb{R}}(f) \in \mathbb{R}^2$.

The **coefficient** embedding τ given by
$\tau(f) = (4, 3)$ and $\tau(g) = (1, -6)$.
Multiplication via convolution product (slow)

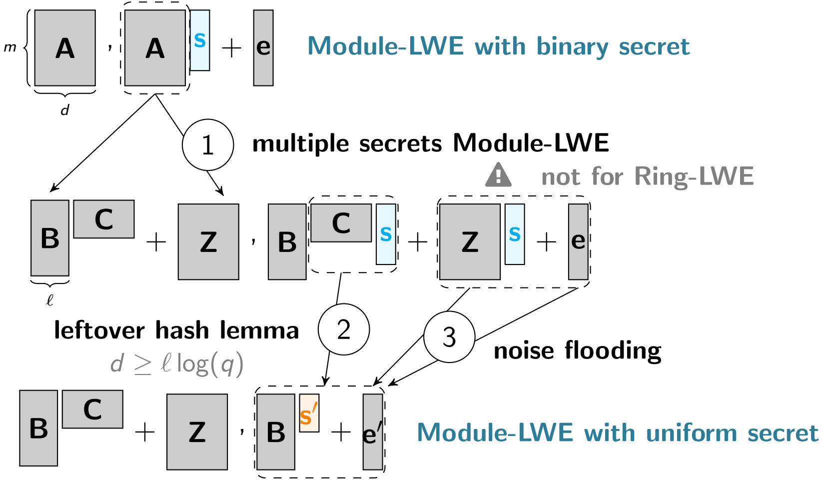
Relation between σ and τ via the **Vandermonde matrix**:

$$\sigma(f) = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \tau(f).$$

Used to speed up computations in Module-LWE.

Step 2: Hardness of binary Module-LWE [GKPV10]

The secret $\mathbf{s} \in R_2^d$ is binary and the secret $\mathbf{s}' \in R_q^\ell$ is modulo q.



Improved noise flooding using Rényi Divergence 1/2

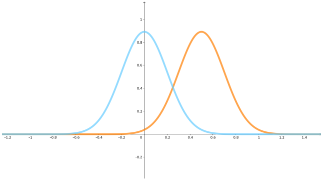
Let P, Q be discrete probability distributions.

In [GKPV10]: Statistical Distance

$$SD(P,Q) = \frac{1}{2} \sum_{x \in \mathrm{Supp}(P)} |P(x) - Q(x)|$$

In our work: Rényi Divergence

$$RD(P,Q) = \sum_{x \in \mathrm{Supp}(P)} \frac{P(x)^2}{Q(x)}$$



Example: two Gaussians D_β and $D_{\beta,s}$,

$$RD(D_\beta, D_{\beta,s}) = \exp\left(\frac{2\pi \|s\|^2}{\beta^2}\right)$$

$$SD(D_\beta, D_{\beta,s}) = \frac{\sqrt{2\pi}\|s\|}{\beta}$$

Improved noise flooding using Rényi Divergence 2/2

Both fulfill the **probability preservation property** for an event E:

$$\begin{array}{llll} \text{[GKPV10]:} & P(E) \leq SD(P,Q) + Q(E) & & \text{(additive)} \\ \textbf{Our work:} & P(E)^2 \leq RD(P,Q) \cdot Q(E) & & \text{(multiplicative)} \end{array}$$

We need: $Q(E)$ negligible $\Rightarrow P(E)$ negligible

Thus: $SD(P,Q) \stackrel{!}{=}$ negligible and $RD(P,Q) \stackrel{!}{=}$ **constant**

Back to example: two Gaussians D_β and $D_{\beta,s}$ with $\|s\| \leq \alpha$

$$\begin{aligned} SD(D_\beta, D_{\beta,s}) &= \frac{\sqrt{2\pi}\|s\|}{\beta} & \Rightarrow \alpha/\beta \leq \text{negligible} \\ RD(D_\beta, D_{\beta,s}) &= \exp\left(\frac{2\pi\|s\|^2}{\beta^2}\right) \approx 1 + \frac{2\pi\|s\|^2}{\beta^2} & \Rightarrow \alpha/\beta \leq \textbf{constant} \end{aligned}$$

(Taylor expansion at 0)

⚠ Rényi Divergence only for search problems.