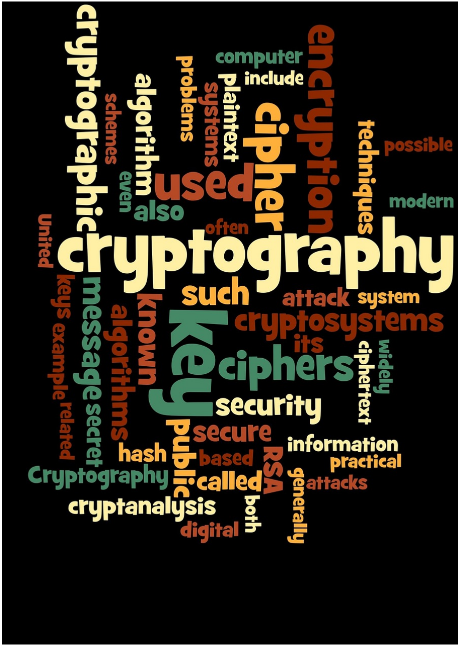
Hardness of Module Learning With Errors With Small Secrets

Katharina Boudgoust Corentin Jeudy
Adeline Roux-Langlois Weiqiang Wen

Univ Rennes, CNRS, IRISA

Séminaire C2 at Inria Paris, 15th October 2021



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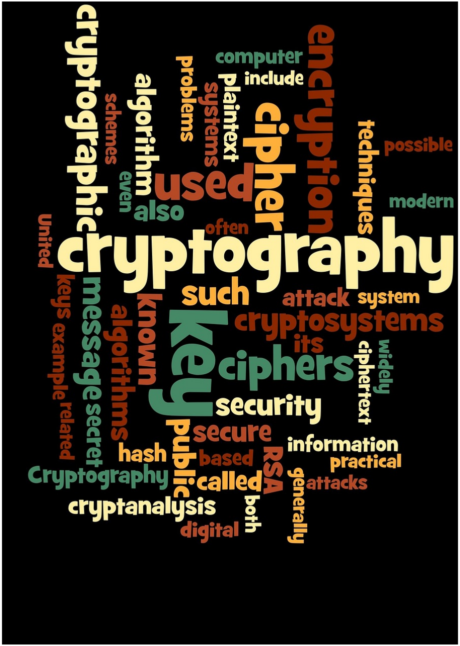

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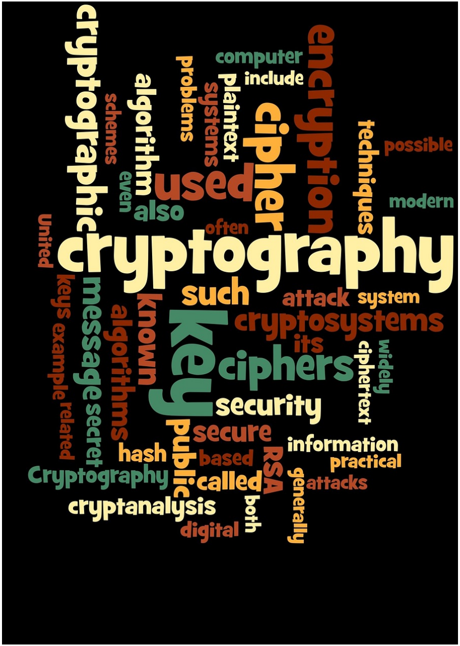

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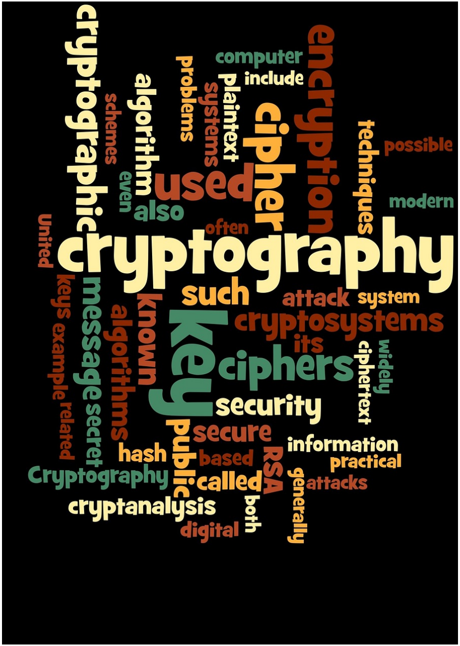

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- Euclidean Lattices
- Codes
- Isogenies
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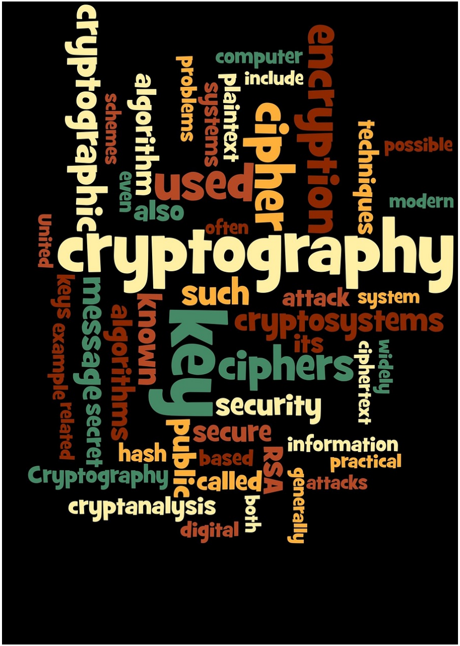

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(Main) Mathematical Problems:

- Short Integer Solution [Ajt96]
- NTRU [HPS98]
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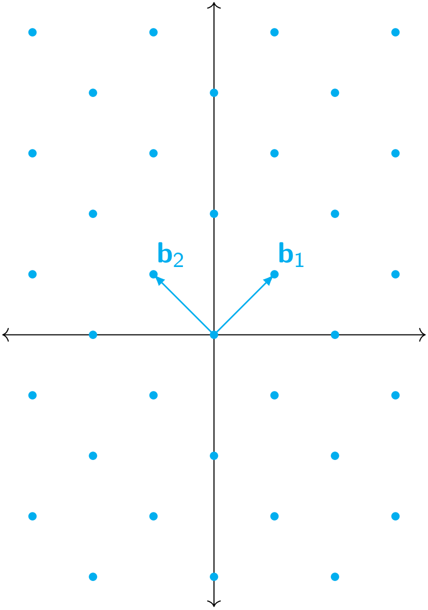

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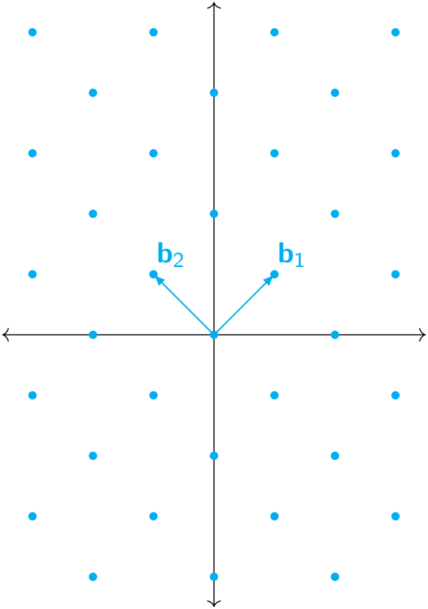
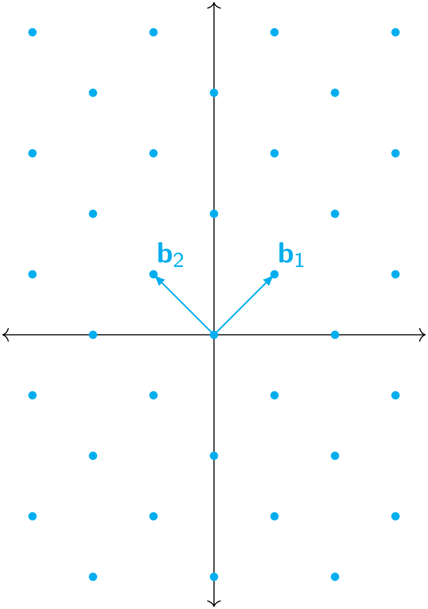


Lattice-Based Cryptography

(Main) Mathematical Problems:

- Short Integer Solution [Ajt96]
- NTRU [HPS98]
- Learning With Errors [Reg05]
  - at least as hard as problems over Euclidean lattices
  - "simple" linear algebra & parallelizable
  - wide range of cryptographic applications
  - in practice: structured variants



Outline

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The Learning With Errors (LWE) Problem

Set $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$ for some integer q

Given $\boxed{A} \sim \mathrm{Unif}(\mathbb{Z}_q^{m \times d})$, $\boxed{b} \in \mathbb{Z}_q^m$, $\boxed{s} \sim$ DistrS over $\mathbb{Z}^d$, $\boxed{e} \sim$ DistrE over $\mathbb{Z}^m$

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⚠ $m(d+1)\log_2 q$ bits

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Reduce sizes of the public key schemes
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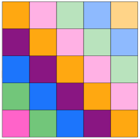
by adding **structure**!



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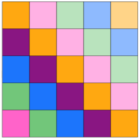

How? Replace $\mathbb{Z}$ by the ring of integers R of some number field K

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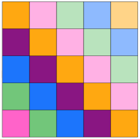

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Before: multiplication of two integers $a \cdot b \in \mathbb{Z}$
Now: multiplication of two polynomials $a \cdot b \in R$ modulo $x^n + 1$

Concrete Example

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\begin{aligned}
f + g &= 2x^3 + 6x^2 - 2x + 8 \\
f \cdot g &= -3x^6 - 10x^5 + 3x^4 + 22x^3 + 8x^2 - 2x + 15 \qquad (\text{use } x^4 + 1 = 0) \\
&= 22x^3 + (3 + 8)x^2 + (10 - 2)x + (-3 + 15) \\
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$\mathrm{Rot}(f)$; depends on R and f

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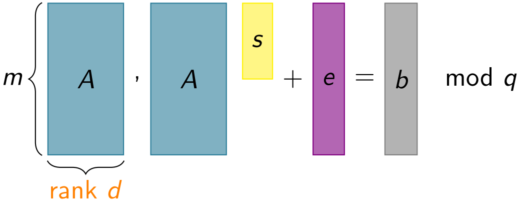

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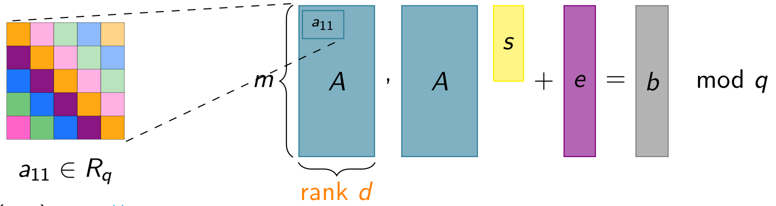

$\mathrm{Rot}(a_{11}) \in \mathbb{Z}_q^{n\times n}$

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Importance of Module-LWE

A majority (5 out of 7) of the finalist candidates for the ongoing NIST standardization process are based on **lattice problems**.

Several among them (3 out of 5) are based on (variants of) **Module-LWE**.

Public Key Encryption

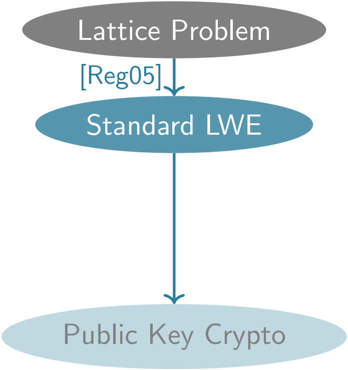
- Crystals-Kyber: Module-LWE
- Saber: Module-LWR (deterministic variant)

Digital Signature

- Crystals-Dilithium: Module-LWE

Overview

Motivation: Theory



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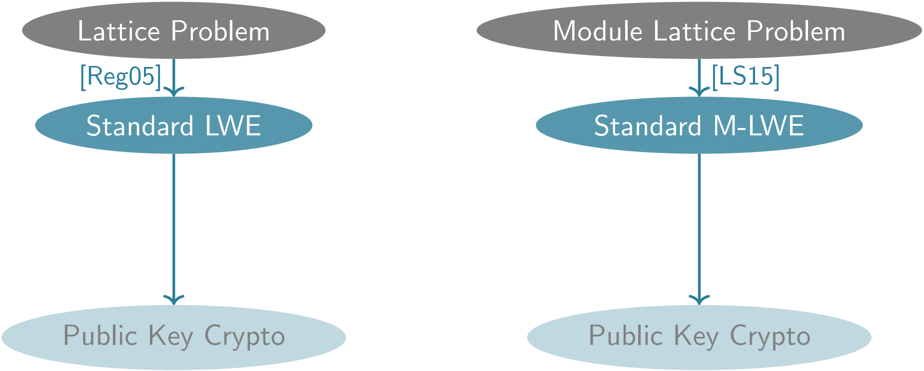
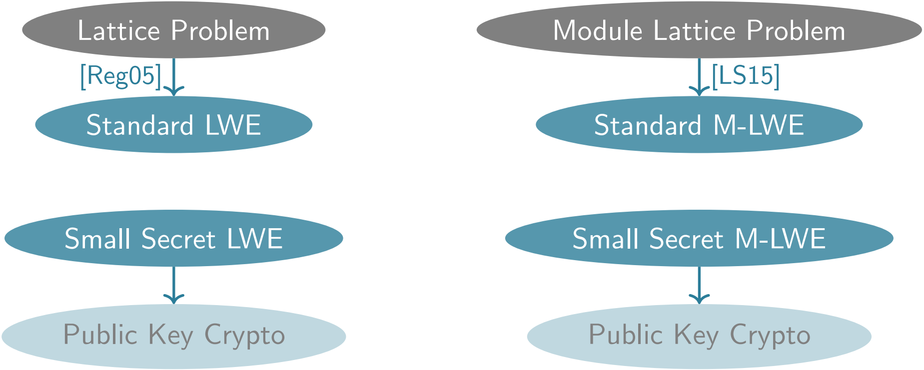


Motivation: Theory vs. Praxis



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- Proof Technique (e.g., Modulus-Rank Switching)

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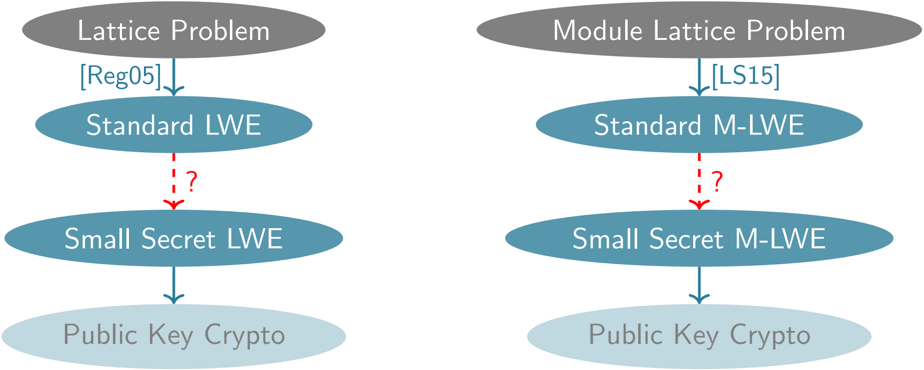

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Hardness of (Module-)LWE with small secrets

Variant	LWE	Module-LWE
Hermite-Normal-Form	[ACPS09]	[ACPS09]
Binary secret	[GKPV10]	
	[BLP$^+$13]	
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η-bounded secret	Generalization of [BLP$^+$13]	

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Our Contributions:

1. Extending and Improving [GKPV10] to M-LWE [BJRW20]
2. Extending [BLP+13] to M-LWE [BJRW21]
3. Generalizing both proofs [Bou21] (not public yet)

Our main result [ia.cr/2020/1020] & [ia.cr/2021/265]

The module learning with errors problem

does not become significantly easier to solve

if the secret is of small norm.

Overview

Hardness of binary Module-LWE (Cyclotomics)

Module-LWE	$\rightarrow$	bin-Module-LWE
modulus q		modulus q
ring degree n		ring degree n
secret $\mathbf{s}'$ mod q		secret $\mathbf{s}$ mod 2
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$\Rightarrow$ **both proofs have their (dis)advantages**

Proof 1: Hardness of binary Module-LWE [GKPV10]

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M-LWE with binary secret

Tikz-Credits to Corentin

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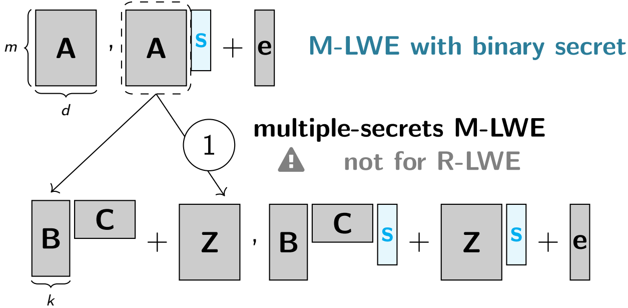

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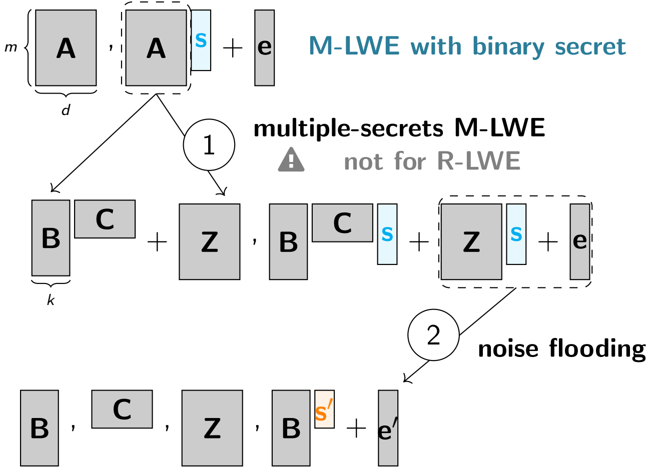

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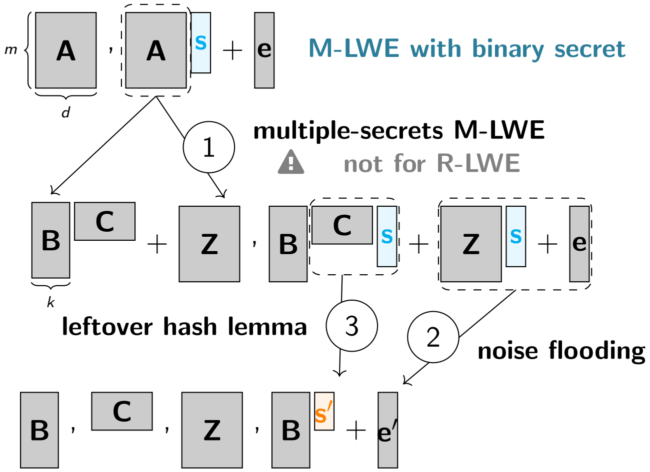

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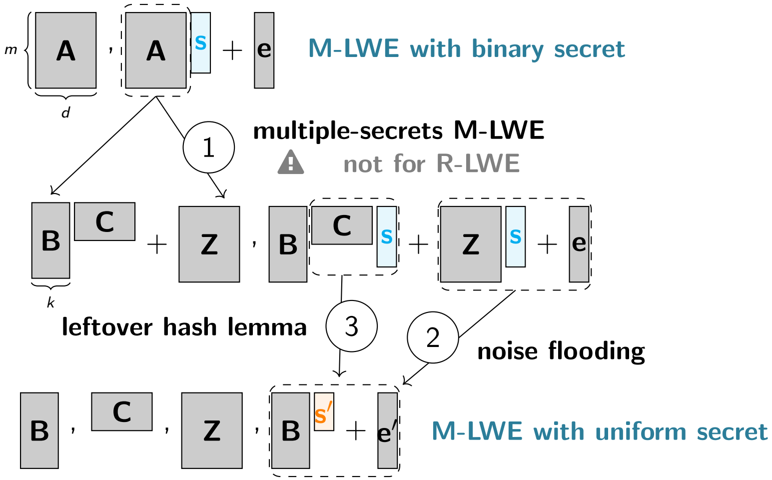

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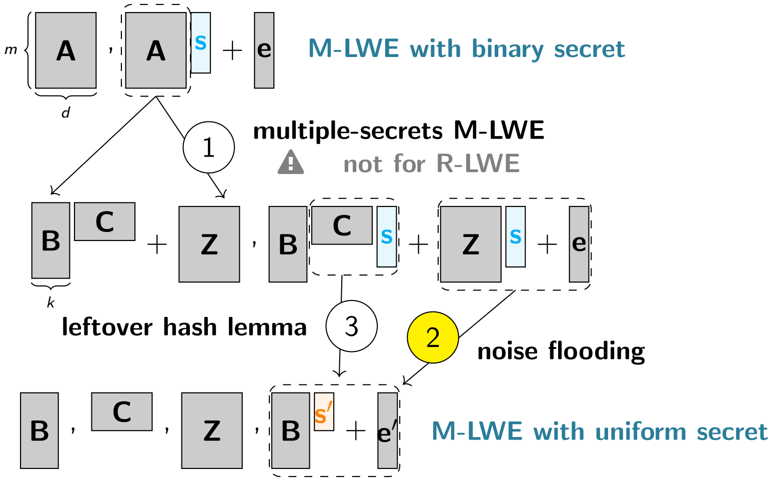


Tikz-Credits to Corentin

Improving 2 by using Rényi Divergence 1/2

Let P, Q be discrete probability distributions.

In [GKPV10]: Statistical Distance

$$SD(P, Q) = \frac{1}{2} \sum_{x \in \mathrm{Supp}(P)} |P(x) - Q(x)|$$

In our work: Rényi Divergence

$$RD(P, Q) = \sum_{x \in \mathrm{Supp}(P)} \frac{P(x)^2}{Q(x)}$$

Improving 2 by using Rényi Divergence 1/2

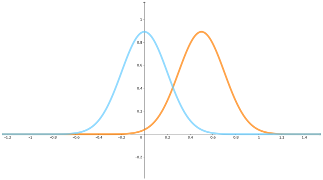
Let P, Q be discrete probability distributions.

In [GKPV10]: Statistical Distance

$$SD(P,Q) = \frac{1}{2} \sum_{x \in \mathrm{Supp}(P)} |P(x) - Q(x)|$$

In our work: Rényi Divergence

$$RD(P,Q) = \sum_{x \in \mathrm{Supp}(P)} \frac{P(x)^2}{Q(x)}$$



Example: two Gaussians D_β and $D_{\beta,s}$,

$$RD(D_\beta, D_{\beta,s}) = \exp\left(\frac{2\pi\|s\|^2}{\beta^2}\right)$$

$$SD(D_\beta, D_{\beta,s}) = \frac{\sqrt{2\pi}\|s\|}{\beta}$$

Improving 2 by using Rényi Divergence 2/2

Both fulfill the **probability preservation property** for an event E:

[GKPV10]:	$P(E)$	$\leq$	$SD(P,Q) + Q(E)$	(additive)
Our work:	$P(E)^2$	$\leq$	$RD(P,Q) \cdot Q(E)$	(multiplicative)

We need: $Q(E)$ negligible $\Rightarrow P(E)$ negligible

Thus: $SD(P,Q) \stackrel{!}{=}$ negligible and $RD(P,Q) \stackrel{!}{=}$ **constant**

Improving 2 by using Rényi Divergence 2/2

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Back to example: two Gaussians D_β and $D_{\beta,s}$ with $\|s\| \leq \alpha$

$$SD(D_\beta, D_{\beta,s}) = \frac{\sqrt{2\pi}\|s\|}{\beta} \qquad \Rightarrow \alpha/\beta \leq \text{negligible}$$

$$RD(D_\beta, D_{\beta,s}) = \exp\left(\frac{2\pi\|s\|^2}{\beta^2}\right) \approx 1 + \frac{2\pi\|s\|^2}{\beta^2} \qquad \Rightarrow \alpha/\beta \leq \textbf{constant}$$

(Taylor expansion at 0)

Improving 2 by using Rényi Divergence 2/2

Both fulfill the **probability preservation property** for an event E:

[GKPV10]:	$P(E)$	$\leq$	$SD(P,Q) + Q(E)$	(additive)
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$$
\begin{aligned}
SD(D_\beta, D_{\beta,s}) &= \frac{\sqrt{2\pi}\|s\|}{\beta} & &\Rightarrow \alpha/\beta \leq \text{negligible} \\
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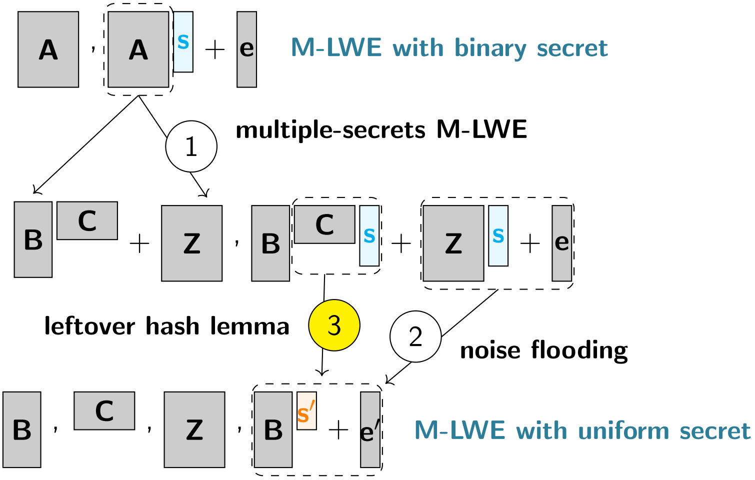
\end{aligned}
$$

(Taylor expansion at 0)

⚠ Rényi Divergence only for search problems.

Proof 1: Hardness of binary Module-LWE [GKPV10]

The secret $\mathbf{s}$ is binary and the secret $\mathbf{s}'$ is modulo q.



Tikz-Credits to Corentin

Improving 3 by using Rényi Divergence

Lemma (leftover hash lemma, adapted from [Mic07])

Let q be prime and let R be the ring of integers of a cyclotomic number field K. Then,

$$SD\left((\mathbf{C}, \mathbf{C}\mathbf{s}), (\mathbf{C}, \mathbf{s}')\right) \leq \frac{1}{2}\sqrt{\left(1 + \frac{q^k}{2^d}\right)^n - 1}, \text{ and}$$
$$RD\left((\mathbf{C}, \mathbf{C}\mathbf{s}), (\mathbf{C}, \mathbf{s}')\right) \leq \left(1 + \frac{q^k}{2^d}\right)^n,$$

where $\mathbf{C} \leftarrow U((R_q)^{k\times d})$, $\mathbf{s} \leftarrow U((R_2)^d)$ *and* $\mathbf{s}' \leftarrow U((R_q)^k)$.

$d \geq k \log_2 q + \omega(\log_2 n) \quad \rightarrow \quad$ SD negligible
$d \geq k \log_2 q + O(\log_2 n) \quad \rightarrow \quad$ RD constant

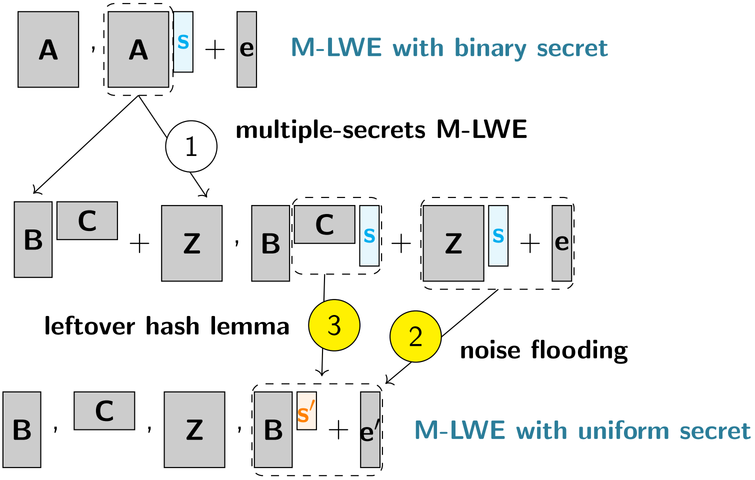
Overview

Question during writing my thesis manuscript:

Module-LWE	$\rightarrow$?	η-Module-LWE
modulus q		modulus q
ring degree n		ring degree n
secret $\mathbf{s}'$ mod q		secret $\mathbf{s}$ mod η
Gaussian width α		Gaussian width β
rank k		rank d

Recall Proof 1 for bin-Module-LWE

The secret $\mathbf{s}$ is binary and the secret $\mathbf{s}'$ is modulo q.



Tikz-Credits to Corentin

Generalizing Step 3

Lemma (leftover hash lemma, adapted from [Mic07])

Let q be prime, $\eta \in \mathbb{N}$ and let R be the ring of integers of a cyclotomic number field K. Then,

$$SD\left((\mathbf{C}, \mathbf{C}\mathbf{s}), (\mathbf{C}, \mathbf{s}')\right) \leq \frac{1}{2}\sqrt{\left(1 + \frac{q^k}{\eta^d}\right)^n - 1}, \text{ and}$$
$$RD\left((\mathbf{C}, \mathbf{C}\mathbf{s}), (\mathbf{C}, \mathbf{s}')\right) \leq \left(1 + \frac{q^k}{\eta^d}\right)^n,$$

where $\mathbf{C} \leftarrow U((R_q)^{k \times d})$, $\mathbf{s} \leftarrow U((R_\eta)^d)$ and $\mathbf{s}' \leftarrow U((R_q)^k)$.

$d \geq k\frac{\log_2 q}{\log_2 \eta} + \omega\left(\frac{\log_2 n}{\log_2 \eta}\right) \quad \rightarrow \quad$ SD negligible

$d \geq k\frac{\log_2 q}{\log_2 \eta} + O\left(\frac{\log_2 n}{\log_2 \eta}\right) \quad \rightarrow \quad$ RD constant

Generalizing to η-bounded secrets (Contribution 3)

Module-LWE	$\rightarrow$	η-Module-LWE
modulus q		modulus q
ring degree n		ring degree n
secret $\mathbf{s}'$ mod q		secret $\mathbf{s}$ mod η
Gaussian width α		Gaussian width β
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Generalizing to η-bounded secrets (Contribution 3)

Module-LWE	$\rightarrow$	η-Module-LWE
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Gaussian width α		Gaussian width β
rank k		rank d

Property	Contribution 1	Contribution 2
LWE analogue	[GKPV10] using RD	[BLP+13]
minimal rank d	$\frac{k \log_2 q}{\log_2 \eta} + O\left(\frac{\log_2 n}{\log_2 \eta}\right)$	$\frac{2k \log_2 q}{\log_2 \eta} + \omega\left(\frac{\log_2 n}{\log_2 \eta}\right)$
noise ratio β/α	$O((\eta - 1)\sqrt{m} n^2 d)$	$O((\eta - 1)^2 n^2 \sqrt{d})$

Generalizing to η-bounded secrets (Contribution 3)

Module-LWE	$\rightarrow$	η-Module-LWE
modulus q		modulus q
ring degree n		ring degree n
secret $\mathbf{s}'$ mod q		secret $\mathbf{s}$ mod η
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LWE analogue	[GKPV10] using RD	[BLP+13]
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noise ratio β/α	$O((\eta - 1)\sqrt{m} n^2 d)$	$O((\eta - 1)^2 n^2 \sqrt{d})$

$\Rightarrow$ **trade-off between minimal rank and noise ratio**

Overview

Hardness of (Module-)LWE with small secrets (Continued)

Variant	LWE	Module-LWE
Hermite-Normal-Form	[ACPS09]	[ACPS09]
Binary secret	[GKPV10]	1
	[BLP+13]	2
	[Mic18]	?
η-bounded secret	Generalization of [BLP+13]	3

Hardness of (Module-)LWE with small secrets (Continued)

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Binary secret	[GKPV10]	1
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η-bounded secret	Generalization of [BLP+13]	3
Entropic secret	[BD20a]	[LWW20] eprint
	[BD20b] Structured-LWE	work in progress

Further work and open questions

Work in progress

- General secret distributions (Entropic M-LWE)
- M-LWE with small noise (extending [MP13])

Open questions **?**

- Smaller rank, in particular rank equals 1 (Ring-LWE)
- Maybe adapting [Mic18] may help?

Further work and open questions

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Thank you.

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