Lattice-Based Cryptography

Criptografía basada en retículos

where to start and where to go next

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Overview of Today's Lecture

Questions we are trying to answer today:

- Part 1: *What are lattices?*
- Part 2: *What are lattice problems?*
- Part 3: *What is lattice-based cryptography?*
- Part 4: *What are the current challenges?*

where to start

whete to go next

References:

- Crash Course Spring 2022 [lecture notes]
- The Lattice Club [link]

Context

👉 The security in public-key cryptography relies on presumably hard mathematical problems.

Currently used problems:

- Discrete logarithm $\rightarrow$ Arantxa's proof system
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Quantum-resistant candidates:

- Codes
- Lattices
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Quantum-resistant candidates:

- Codes
- Lattices → now
- Isogenies → later with Chloe
- Multivariate systems
- **?**

Fernando (INCA)

US National Institute of Standards and Technology (NIST) Project

- 2016: start of NIST's post-quantum cryptography project[*]
- 2022: selection of 4 schemes, 3 of them relying on lattice problems

Public Key Encryption:

- Kyber

Digital Signature:

- Dilithium
- Falcon
- SPHINCS+

Lattice-based cryptography plays a leading role in designing post-quantum cryptography.

[*] https://csrc.nist.gov/projects/post-quantum-cryptography

Part 1:

What is a lattice?

Euclidean Lattices

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- additive subgroup: $\mathbf{0} \in \Lambda$, and for all $\mathbf{x}, \mathbf{y} \in \Lambda$ it holds $\mathbf{x} + \mathbf{y}, -\mathbf{x} \in \Lambda$;
- discrete: every $\mathbf{x} \in \Lambda$ has a neighborhood in which $\mathbf{x}$ is the only lattice point.

  $\exists \varepsilon > 0$ such that $\mathcal{B}(\mathbf{x}, \varepsilon) \cap \Lambda = \{ \mathbf{x} \}$

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There exists a finite basis $\mathbf{B} = (\mathbf{b}_1, \ldots, \mathbf{b}_n) \subset \mathbb{R}^n$ such that

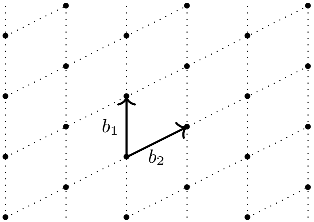
$$\Lambda(\mathbf{B}) = \left\{ \sum_{i=1}^{n} z_i \mathbf{b}_i \colon z_i \in \mathbb{Z} \right\}.$$

- n is the rank of Λ

Euclidean Lattices

Let $\mathbf{B} \in \mathbb{R}^{n \times n}$ be a basis for Λ, i.e.,

$$\Lambda(\mathbf{B}) = \left\{ \sum_{i=1}^{n} z_i \mathbf{b}_i \colon z_i \in \mathbb{Z} \right\} = \{\mathbf{Bz} \colon \mathbf{z} \in \mathbb{Z}^n\}.$$



$\Lambda \in \mathbb{R}^2$

Euclidean Lattices

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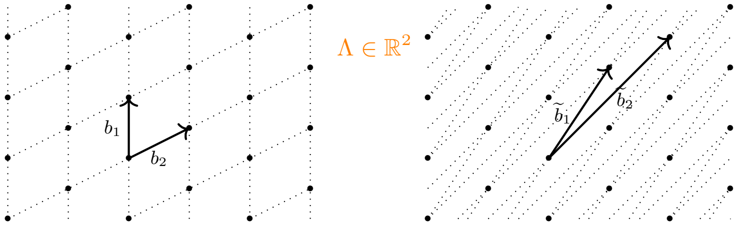
$$\Lambda(\mathbf{B}) = \left\{ \sum_{i=1}^{n} z_i \mathbf{b}_i \colon z_i \in \mathbb{Z} \right\} = \{\mathbf{B}\mathbf{z} \colon \mathbf{z} \in \mathbb{Z}^n\}\,.$$



- $\mathbf{U} \in \mathbb{Z}^{n \times n}$ unimodular, then $\widetilde{\mathbf{B}} = \mathbf{B} \cdot \mathbf{U}$ also a basis of Λ $\qquad$ $\det(\mathbf{U}) = \pm 1$
- $\det(\Lambda) := |\det(\mathbf{B})|$

Dual Lattices

The **dual** of a lattice $\Lambda \subset \mathbb{R}^n$ is defined as

$$\Lambda^\vee = \{\mathbf{w} \in \mathbb{R}^n \colon \langle \mathbf{w}, \mathbf{x} \rangle \in \mathbb{Z} \ \forall \mathbf{x} \in \Lambda\} .$$

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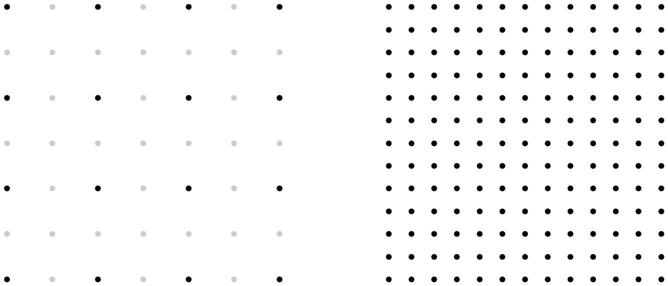
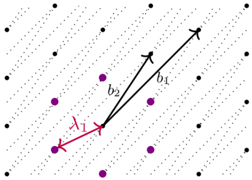
- if $\mathbf{B}$ a basis for Λ, then $(\mathbf{B}^T)^{-1}$ a basis for $\Lambda^\vee$
- $\det(\Lambda^\vee) = \det(\Lambda)^{-1}$



$2\mathbb{Z}^2$ and its dual $\frac{1}{2}\mathbb{Z}^2$

Lattice Minimum & Special Lattices



The **minimum** of a lattice $\Lambda \subset \mathbb{R}^n$ is defined as

$$\lambda_1(\Lambda) = \min_{\mathbf{x} \in \Lambda \setminus \{\mathbf{0}\}} \|\mathbf{x}\|_2.$$

- Minkowski: $\lambda_1(\Lambda) \leq \sqrt{n} \cdot \det(\Lambda)^{1/n}$
- Exercise: $\lambda_1(\Lambda) \cdot \lambda_1(\Lambda^\vee) \leq n$

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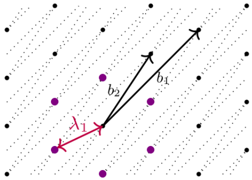

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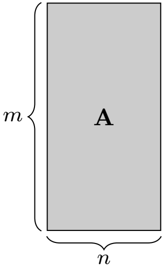
- Exercise: $\lambda_1(\Lambda) \cdot \lambda_1(\Lambda^\vee) \leq n$

Let $\mathbf{A} \in \mathbb{Z}_q^{m \times n}$ for some $n, m, q \in \mathbb{N}$ with $n \leq m$

$\mathbb{Z}_q$ integers modulo q

$$\Lambda_q(\mathbf{A}) = \{\mathbf{y} \in \mathbb{Z}^m : \mathbf{y} = \mathbf{A}\mathbf{s} \bmod q \text{ for some } \mathbf{s} \in \mathbb{Z}^n\}$$
$$\Lambda_q^\perp(\mathbf{A}) = \left\{\mathbf{y} \in \mathbb{Z}^m : \mathbf{A}^T\mathbf{y} = \mathbf{0} \bmod q\right\}$$



- Exercise: $\Lambda_q^\perp(\mathbf{A}) = q \cdot \Lambda_q(\mathbf{A})^\vee$

Part 2:

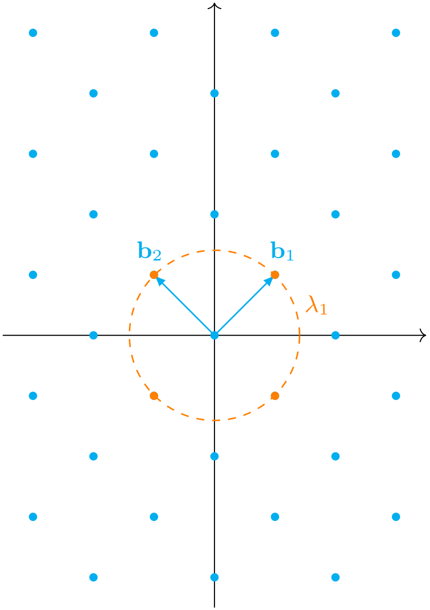
What are lattice problems?

Shortest Vector Problem

Given a lattice $\Lambda \in \mathbb{R}^n$ of rank n.

The **shortest vector problem** (SVP) asks to find a vector $\mathbf{w} \in \Lambda$ such that

$$\|\mathbf{w}\|_2 = \lambda_1(\Lambda).$$

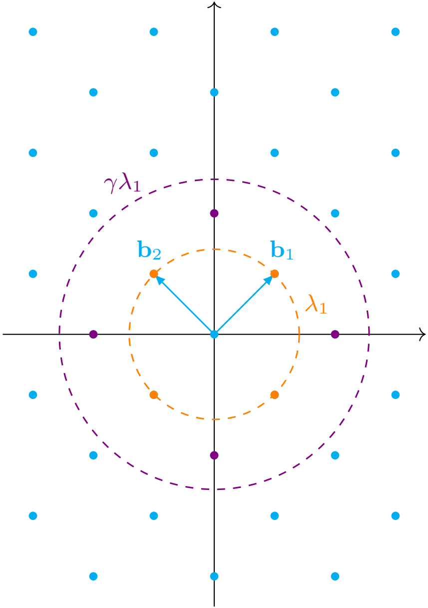


Shortest Vector Problem

Given a lattice $\Lambda \in \mathbb{R}^n$ of rank n.

The **approximate shortest vector problem** (SVP$_\gamma$) for $\gamma \geq 1$ asks to find a vector $\mathbf{w} \in \Lambda$ such that

$$\|\mathbf{w}\|_2 \leq \gamma \lambda_1(\Lambda).$$



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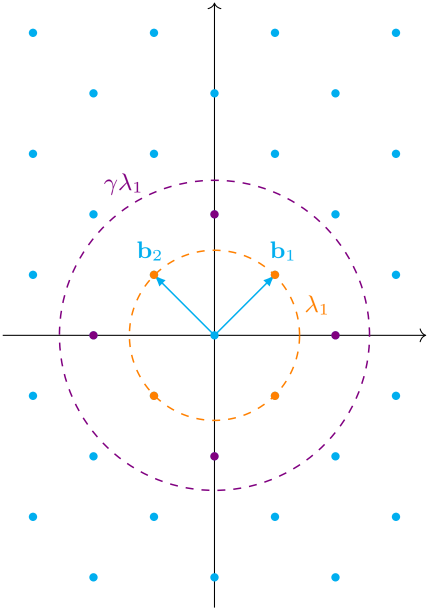
The complexity of SVP$_\gamma$ increases with n, but decreases with γ.

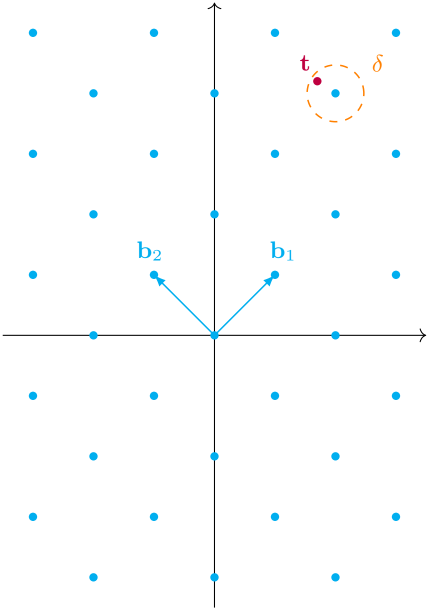
Conjecture:

There is no polynomial-time classical or quantum algorithm that solves SVP$_\gamma$ to within polynomial factors.



Bounded Distance Decoding

Given a lattice $\Lambda \in \mathbb{R}^n$ of rank n and a target $\mathbf{t} \in \mathbb{R}^n$ such $\text{dist}(\Lambda, \mathbf{t}) \leq \delta < \lambda_1(\Lambda)$.



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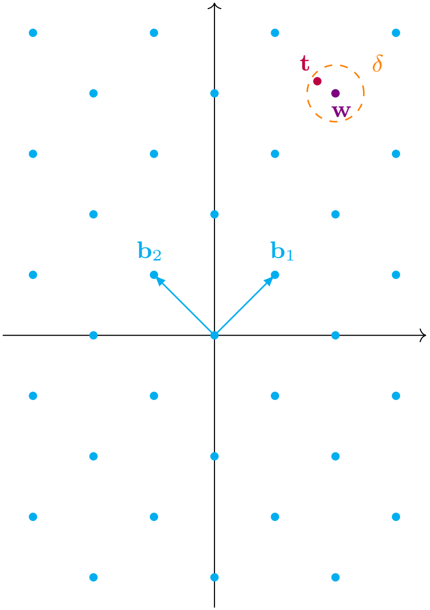
The **bounded distance decoding** (BDD_δ) problem asks to find the unique vector $\mathbf{w} \in \Lambda$ such that

$$\|\mathbf{w} - \mathbf{t}\|_2 \leq \delta.$$



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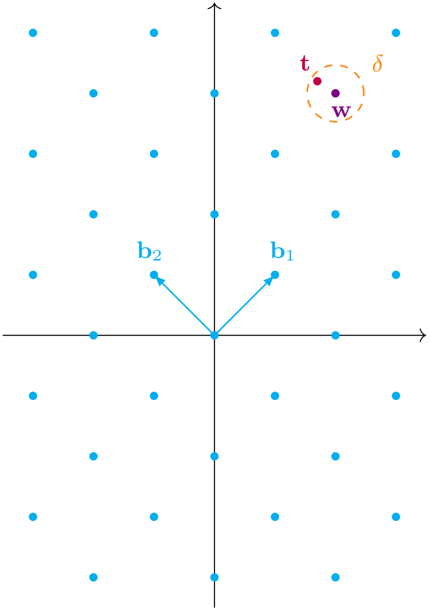
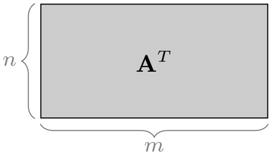


Short Integer Solution [Ajt96]

Given a matrix $\mathbf{A} \in \mathbb{Z}_q^{m \times n}$ sampled uniformly at random and bound $\beta > 0$.



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The **short integer solution** (SIS_β) problem asks to find a vector $\mathbf{z} \in \mathbb{Z}^m$ of norm $0 < \|\mathbf{z}\|_2 \leq \beta$ such that

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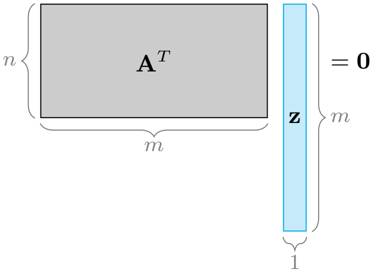

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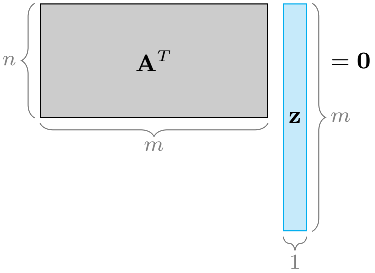

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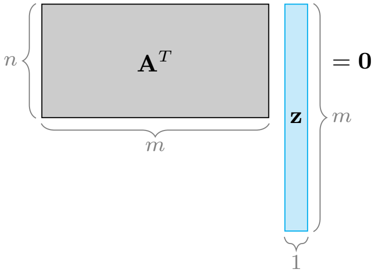

⚠ The norm restriction makes it a hard problem!

Recall:

$$\Lambda_q^\perp(\mathbf{A}) = \left\{ \mathbf{y} \in \mathbb{Z}^m : \mathbf{A}^T \mathbf{y} = \mathbf{0} \bmod q \right\}$$

☞ SIS$_\beta$ equals SVP$_\gamma$ in the special lattice $\Lambda_q^\perp(\mathbf{A})$ for $\beta = \gamma \cdot \lambda_1(\Lambda_q^\perp(\mathbf{A}))$

Learning With Errors [Reg05]

Given a matrix $\mathbf{A} \leftarrow \mathsf{Unif}(\mathbb{Z}_q^{m \times n})$.

Given a vector $\mathbf{b} \in \mathbb{Z}_q^m$, where $\mathbf{b} = \mathbf{A}\mathbf{s} + \mathbf{e} \bmod q$ for

- secret $\mathbf{s} \in \mathbb{Z}_q^n$ sampled from distribution D_s and
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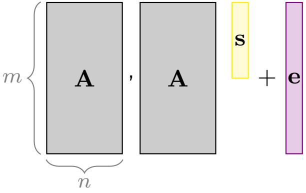

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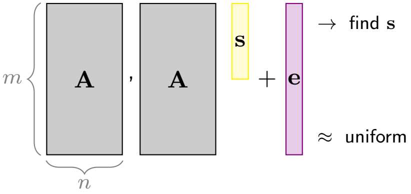
Search learning with errors (S-LWE$_\delta$) asks to find $\mathbf{s}$.

Decision learning with errors (D-LWE$_\delta$) asks to distinguish $(\mathbf{A}, \mathbf{b})$ from the uniform distribution over $\mathbb{Z}_q^{m \times n} \times \mathbb{Z}_q^m$.



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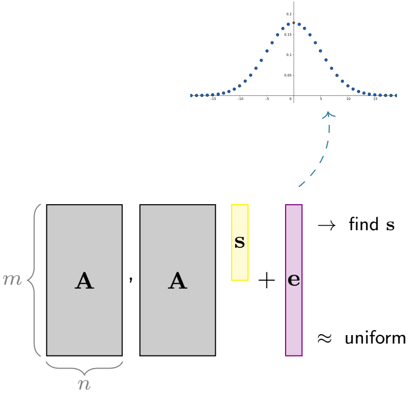

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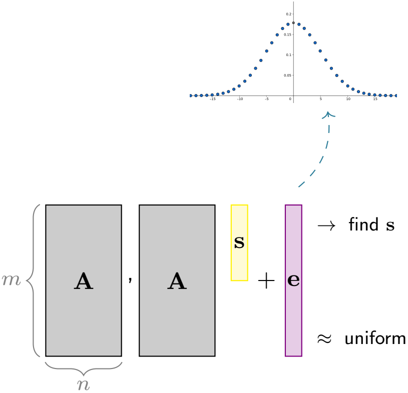

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Exercise: S-LWE$_\delta$ equals BDD$_\delta$ in the special lattice $\Lambda_q(\mathbf{A})$.

Connection between LWE and SIS

☞ If there is an efficient solver for SIS_β, then there is an efficient solver for $\mathsf{D\text{-}LWE}_\delta$, assuming $\delta \cdot \beta \ll q$.

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Proof.

Given $(\mathbf{A}, \mathbf{b})$, our goal is to decide whether 1) $\mathbf{b} = \mathbf{A}\mathbf{s} + \mathbf{e}$ for $\|\mathbf{e}\|_2 \leq \delta$ or 2) $\mathbf{b} \leftarrow \mathsf{Unif}(\mathbb{Z}_q^m)$.

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Forward $\mathbf{A}$ to SIS-solver and receive back $\mathbf{z}$ such that $\mathbf{A}^T\mathbf{z} = \mathbf{0} \bmod q$ and $\|\mathbf{z}\|_2 \leq \beta$.

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Compute $\left\|\mathbf{b}^T\mathbf{z}\right\|_\infty$. If the norm is $\ll q$, claim that we are in case 1). Else, claim that we are in case 2).

□

Connection between LWE and SIS

If there is an efficient solver for SIS$_\beta$, then there is an efficient solver for D-LWE$_\delta$, assuming $\delta \cdot \beta \ll q$.

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Forward $\mathbf{A}$ to SIS-solver and receive back $\mathbf{z}$ such that $\mathbf{A}^T\mathbf{z} = \mathbf{0} \bmod q$ and $\|\mathbf{z}\|_2 \leq \beta$.

Compute $\left\|\mathbf{b}^T\mathbf{z}\right\|_\infty$. If the norm is $\ll q$, claim that we are in case 1). Else, claim that we are in case 2).

Case 1) $\mathbf{b} = \mathbf{A}\mathbf{s} + \mathbf{e}$, thus $\mathbf{b}^T\mathbf{z} = \mathbf{s}^T\mathbf{A}^T\mathbf{z} + \mathbf{e}^T\mathbf{z} = \mathbf{e}^T\mathbf{z} \bmod q$. Thus $\left\|\mathbf{b}^T\mathbf{z}\right\|_\infty \leq \left\|\mathbf{e}^T\right\|_\infty \cdot \|\mathbf{z}\|_\infty \leq \delta \cdot \beta \ll q$.

Case 2) $\mathbf{b}$ uniform, so is $\mathbf{b}^T\mathbf{z}$ and hence $\left\|\mathbf{b}^T\mathbf{z}\right\|_\infty$ with high chances larger than $\delta\beta$. □

Part 3:

What is lattice-based cryptography?

Collision-Resistant Hash Function from SIS [Ajt96]

A function $f \colon$ Domain $\rightarrow$ Range is called **collision-resistant** if it is hard to output two elements $\mathbf{x}, \mathbf{x}' \in$ Domain such that

$$f(\mathbf{x}) = f(\mathbf{x}') \text{ and } \mathbf{x} \neq \mathbf{x}'.$$

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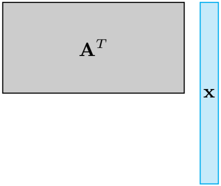
Set $f_{\mathbf{A}}\colon \{0,1\}^m \to \mathbb{Z}_q^n$ with $f_{\mathbf{A}}(\mathbf{x}) = \mathbf{A}^T\mathbf{x} \bmod q$ for $\mathbf{A} \leftarrow \text{Unif}(\mathbb{Z}_q^{m \times n})$.



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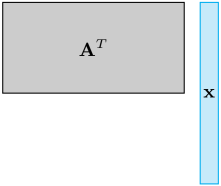


Exercise: Assuming SIS is hard to solve for $\beta = \sqrt{m}$, then $f_{\mathbf{A}}$ is collision-resistant

Hint: $\mathbf{x} \neq \mathbf{x}' \in \{0,1\}^m \Leftrightarrow \mathbf{0} \neq \mathbf{x} - \mathbf{x}' \in \{-1,0,1\}^m$

$\mathbf{A}^T\mathbf{x} = \mathbf{A}^T\mathbf{x}' \Leftrightarrow \mathbf{A}^T(\mathbf{x} - \mathbf{x}') = 0$

Reminder: Public-Key Encryption (PKE)

A public-key encryption scheme $\Pi = (\mathsf{KGen}, \mathsf{Enc}, \mathsf{Dec})$ consists of three algorithms:

- $\mathsf{KGen}(1^\lambda) \to (\mathsf{sk}, \mathsf{pk})$ — λ security parameter
- $\mathsf{Enc}(\mathsf{pk}, m) \to \mathsf{ct}$
- $\mathsf{Dec}(\mathsf{sk}, \mathsf{ct}) = m'$

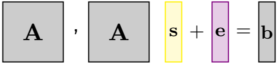
Correctness: $\mathsf{Dec}(\mathsf{sk}, \mathsf{Enc}(\mathsf{pk}, m)) = m$ during an honest execution

Semantic Security: $\mathsf{Enc}(\mathsf{pk}, m_0)$ is indistinguishable from $\mathsf{Enc}(\mathsf{pk}, m_1)$
(IND-CPA)

Public-Key Encryption from LWE [Reg05]

Let χ be distribution on $\mathbb{Z}$.

- KGen(1^λ):
  - $\mathbf{A} \leftarrow$ Unif$(\mathbb{Z}_q^{n \times n})$ and $\mathbf{s}, \mathbf{e} \leftarrow \chi^n$
  - $\mathbf{b} = \mathbf{A}\mathbf{s} + \mathbf{e} \bmod q$
  - Output sk $= \mathbf{s}$ and pk $= (\mathbf{A}, \mathbf{b})$



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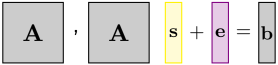
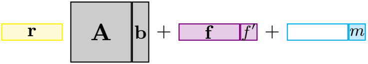
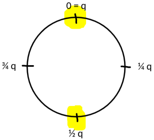


- $\mathsf{Enc}(\mathsf{pk}, m \in \{0,1\})$:
  - $\mathbf{r}, \mathbf{f} \leftarrow \chi^n$ and $f' \leftarrow \chi$
  - $\mathbf{u} = \mathbf{r}\mathbf{A} + \mathbf{f}$
  - $v = \mathbf{r}\mathbf{b} + f' + \lfloor q/2 \rfloor \cdot m$
  - Output $\mathsf{ct} = (\mathbf{u}, v)$





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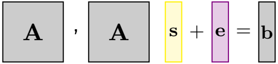

- Enc$(\mathsf{pk}, m \in \{0,1\})$:
  - $\mathbf{r}, \mathbf{f} \leftarrow \chi^n$ and $f' \leftarrow \chi$
  - $\mathbf{u} = \mathbf{r}\mathbf{A} + \mathbf{f}$
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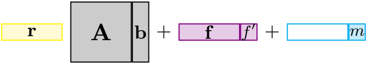
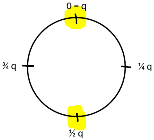
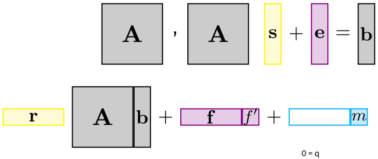


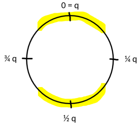
- Dec(sk, ct):
  - If $v - \mathbf{u}\mathbf{s}$ is closer to 0 than to $q/2$, output $m' = 0$
  - Else output $m' = 1$



Public-Key Encryption from LWE [Reg05]

- KGen(1^λ):
  - $\mathbf{A} \leftarrow$ Unif($\mathbb{Z}_q^{n\times n}$) and $\mathbf{s}, \mathbf{e} \leftarrow \chi^n$
  - $\mathbf{b} = \mathbf{As} + \mathbf{e} \bmod q$
  - Output sk $= \mathbf{s}$ and pk $= (\mathbf{A}, \mathbf{b})$
- Enc(pk, $m \in \{0,1\}$):
  - $\mathbf{r}, \mathbf{f} \leftarrow \chi^n$ and $f' \leftarrow \chi$
  - $\mathbf{u} = \mathbf{rA} + \mathbf{f}$
  - $v = \mathbf{rb} + f' + \lfloor q/2 \rfloor \cdot m$
  - Output ct $= (\mathbf{u}, v)$
- Dec(sk, ct):
  - If $v - \mathbf{us}$ is closer to 0 than to $q/2$, output $m' = 0$
  - Else output $m' = 1$





Correctness:

$$v - \mathbf{us} = \mathbf{r}(\mathbf{As} + \mathbf{e}) + f' + \lfloor q/2 \rfloor \cdot m - (\mathbf{rA} + \mathbf{f})\mathbf{s}$$
$$= \underbrace{\mathbf{re} + f' - \mathbf{fs}}_{* \text{ ciphertext noise}} + \lfloor q/2 \rfloor m$$

Decryption succeeds if $|*| < q/8$

Public-Key Encryption from LWE [Reg05]

- KGen(1^λ):
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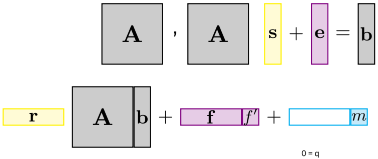

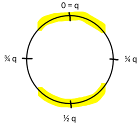

Correctness: Let χ be B-bounded with $2nB^2 + B < q/8$

$$v - \mathbf{u}\mathbf{s} = \mathbf{r}(\mathbf{A}\mathbf{s} + \mathbf{e}) + f' + \lfloor q/2 \rfloor \cdot m - (\mathbf{r}\mathbf{A} + \mathbf{f})\mathbf{s}$$
$$= \underbrace{\mathbf{r}\mathbf{e} + f' - \mathbf{f}\mathbf{s}}_{*\ \text{ciphertext noise}} + \lfloor q/2 \rfloor m$$

Decryption succeeds if $|*| < q/8$

$$|*| = |\mathbf{r}\mathbf{e} + f' - \mathbf{f}\mathbf{s}| \leq \|\mathbf{r}\|_2 \cdot \|\mathbf{e}\|_2 + \|\mathbf{f}\|_2 \cdot \|\mathbf{s}\|_2 + |f'| \leq 2(\sqrt{n}B \cdot \sqrt{n}B) + B < q/8$$

Public-Key Encryption from LWE [Reg05]

- KGen(1^λ):
  - $\mathbf{A} \leftarrow \mathsf{Unif}(\mathbb{Z}_q^{n\times n})$ and $\mathbf{s}, \mathbf{e} \leftarrow \chi^n$
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  - Output $\mathsf{sk} = \mathbf{s}$ and $\mathsf{pk} = (\mathbf{A}, \mathbf{b})$
- Enc$(\mathsf{pk}, m \in \{0,1\})$:
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  - $\mathbf{u} = \mathbf{r}\mathbf{A} + \mathbf{f}$
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  - Output $\mathsf{ct} = (\mathbf{u}, v)$
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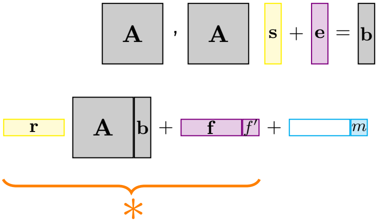

Semantic Security: Assume hardness of decision LWE

1. replace $\mathbf{b}$ by uniform random vector
2. replace non-message part ($*$) by uniform random vector
3. then the message is completely hidden

Kyber - Selected for Standardization by NIST

Kyber = the previous construction + several improvements

Main improvements:

1. Structured LWE variant (most important)
2. LWE secret and noise from centered binomial distribution
3. Pseudorandomness for distributions
4. Ciphertext compression

Sources:

- Website of Kyber: https://pq-crystals.org/kyber/
- Latest specifications [link]
- Tutorial by V. Lyubashevsky [link]

5 Min

Part 4:

What are (my) current challenges?

Re-Reminder: Public Key Encryption (PKE)

PKE scheme:

- $\mathsf{KGen}(1^\lambda) \to (\mathsf{pk}, \mathsf{sk})$ — λ security parameter
- $\mathsf{Enc}(\mathsf{pk}, m) \to \mathsf{ct}$
- $\mathsf{Dec}(\mathsf{sk}, \mathsf{ct}) \to m'$

Properties:

- Correctness
- Semantic security

Re-Reminder: Public Key Encryption (PKE)

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- $\mathsf{KGen}(1^\lambda) \to (\mathsf{pk}, \mathsf{sk})$ — λ security parameter
- $\mathsf{Enc}(\mathsf{pk}, m) \to \mathsf{ct}$
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Properties:

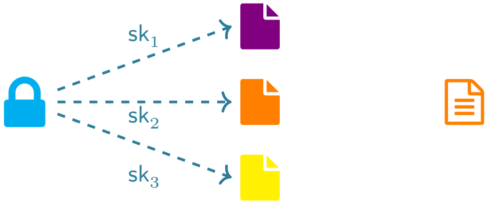
- Correctness
- Semantic security

⚠ Single Point of Failure

Threshold Public Key Encryption (TPKE)

t-out-of-n Threshold PKE scheme:

- $\mathsf{KGen}(1^\lambda) \to (\mathsf{pk}, \mathsf{sk}_1, \ldots, \mathsf{sk}_n)$ secret sharing
- $\mathsf{Enc}(\mathsf{pk}, m) \to \mathsf{ct}$
- $\mathsf{PartDec}(\mathsf{sk}_i, \mathsf{ct}') \to d_i$ $i \in \{1, \ldots, n\}$
- $\mathsf{Combine}(\{d_i\}_{i \in S}) \to m'$ $S \subset \{1, \ldots, n\}$



* https://csrc.nist.gov/projects/threshold-cryptography

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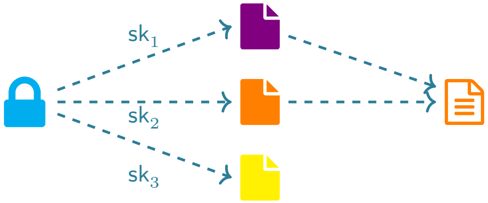
- Combine$(\{d_i\}_{i \in S}) \to m'$ — $S \subset \{1, \ldots, n\}$



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Properties:

- Correctness — for $|S| > t$ recover correct message
- Partial decryption security — for $|S| \leq t$ no information is leaked
- Semantic security

Applications:

- Storing sensitive data — NIST's call*
- Electronic voting protocols
- Multiparty computations — $\to$ Chris yesterday, Daniel later

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Reminder: PKE from LWE

- $\mathsf{KGen}(1^\lambda)$:
  - $\mathbf{A} \leftarrow \mathsf{Unif}(\mathbb{Z}_q^{n \times n})$ and $\mathbf{s}, \mathbf{e} \leftarrow \chi^n$
  - $\mathbf{b} = \mathbf{A}\mathbf{s} + \mathbf{e} \bmod q$
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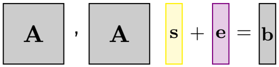
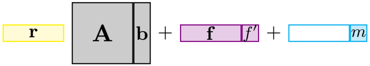
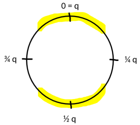


- $\mathsf{Enc}(\mathsf{pk}, m \in \{0, 1\})$:
  - $\mathbf{r}, \mathbf{f} \leftarrow \chi^n$ and $f' \leftarrow \chi$
  - $\mathbf{u} = \mathbf{r}\mathbf{A} + \mathbf{f}$
  - $v = \mathbf{r}\mathbf{b} + f' + \lfloor q/2 \rfloor \cdot m$
  - Output $\mathsf{ct} = (\mathbf{u}, v)$



- $\mathsf{Dec}(\mathsf{sk}, \mathsf{ct})$:
  - If $v - \mathbf{u}\mathbf{s}$ is closer to 0 than to $q/2$, output $m' = 0$
  - Else output $m' = 1$



Reminder: PKE from LWE

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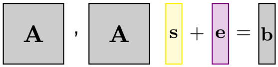
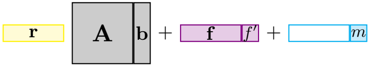


- Enc(pk, $m \in \{0,1\}$):
  - $\mathbf{r}, \mathbf{f} \leftarrow \chi^n$ and $f' \leftarrow \chi$
  - $\mathbf{u} = \mathbf{r}\mathbf{A} + \mathbf{f}$
  - $v = \mathbf{r}\mathbf{b} + f' + \lfloor q/2 \rfloor \cdot m$
  - Output ct $= (\mathbf{u}, v)$



- Dec(sk, ct):
  - If $v - \mathbf{u}\mathbf{s}$ is closer to 0 than to $q/2$, output $m' = 0$
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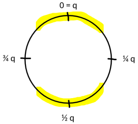


In order to thresholdize it:

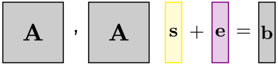
modify KGen and replace Dec by PartDec and Combine

(Enc stays the same)

Full-Threshold PKE from LWE, First Trial

(n-out-of-n)

- KGen(1^λ):
  - $\mathbf{A} \leftarrow$ Unif$(\mathbb{Z}_q^{n\times n})$ and $\mathbf{s}, \mathbf{e} \leftarrow \chi^n$
  - $\mathbf{b} = \mathbf{A}\mathbf{s} + \mathbf{e} \bmod q$
  - $\mathbf{s}_1, \ldots, \mathbf{s}_{n-1} \leftarrow$ Unif$(\mathbb{Z}_q^n)$
  - $\mathbf{s}_n = \mathbf{s} - \sum_{i=1}^{n-1} \mathbf{s}_i$
  - Output $\mathsf{sk}_i = \mathbf{s}_i$ and $\mathsf{pk} = (\mathbf{A}, \mathbf{b})$



- PartDec$(\mathsf{sk}_i, (\mathbf{u}, v))$:
  - Output $d_i = \mathbf{u}\mathbf{s}_i$

- Combine$(d_1, \ldots, d_n)$:
  - $d = \sum_{i=1}^{n} d_i$
  - If $v - d$ is closer to 0 than to $q/2$, output $m' = 0$
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Full-Threshold PKE from LWE, First Trial

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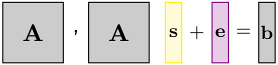

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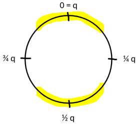

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  - If $v - d$ is closer to 0 than to $q/2$, output $m' = 0$
  - Else output $m' = 1$

Correctness: given $d_1, \ldots, d_n$

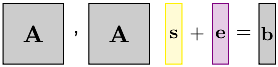
$$v - \sum_{i=1}^{n} \mathbf{u}\mathbf{s}_i = v - \mathbf{u}\sum_{i=1}^{n} \mathbf{s}_i = v - \mathbf{u}\mathbf{s}$$

$$= \underbrace{\mathbf{r}\mathbf{e} + f' - \mathbf{f}\mathbf{s}}_{*\ \text{ciphertext noise}} + \lfloor q/2 \rfloor m$$

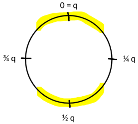
Decryption succeeds if $|*| < q/8$

Full-Threshold PKE from LWE, First Trial

- KGen(1^λ):
  - $\mathbf{A} \leftarrow$ Unif$(\mathbb{Z}_q^{n \times n})$ and $\mathbf{s}, \mathbf{e} \leftarrow \chi^n$
  - $\mathbf{b} = \mathbf{A}\mathbf{s} + \mathbf{e} \bmod q$
  - $\mathbf{s}_1, \ldots, \mathbf{s}_{n-1} \leftarrow$ Unif$(\mathbb{Z}_q^n)$
  - $\mathbf{s}_n = \mathbf{s} - \sum_{i=1}^{n-1} \mathbf{s}_i$
  - Output $\mathsf{sk}_i = \mathbf{s}_i$ and $\mathsf{pk} = (\mathbf{A}, \mathbf{b})$



- PartDec$(\mathsf{sk}_i, (\mathbf{u}, v))$:
  - Output $d_i = \mathbf{u}\mathbf{s}_i$



- Combine$(d_1, \ldots, d_n)$:
  - $d = \sum_{i=1}^n d_i$
  - If $v - d$ is closer to 0 than to $q/2$, output $m' = 0$
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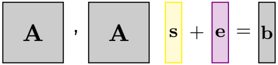
$$v - \sum_{i=1}^{n} \mathbf{u}\mathbf{s}_i = v - \mathbf{u}\sum_{i=1}^{n} \mathbf{s}_i = v - \mathbf{u}\mathbf{s}$$

$$= \underbrace{\mathbf{r}\mathbf{e} + f' - \mathbf{f}\mathbf{s}}_{* \text{ ciphertext noise}} + \lfloor q/2 \rfloor m$$

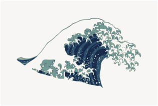
⚠ But $(*)$ leaks information about $\mathsf{sk} = \mathbf{s}$!

Full-Threshold PKE from LWE [BD10]

- $\mathsf{KGen}(1^\lambda)$:
  - $\mathbf{A} \leftarrow \mathsf{Unif}(\mathbb{Z}_q^{n \times n})$ and $\mathbf{s}, \mathbf{e} \leftarrow \chi^n$
  - $\mathbf{b} = \mathbf{A}\mathbf{s} + \mathbf{e} \bmod q$
  - $\mathbf{s} = \sum_{i=1}^{n} \mathbf{s}_i$
  - Output $\mathsf{sk}_i = \mathbf{s}_i$ and $\mathsf{pk} = (\mathbf{A}, \mathbf{b})$



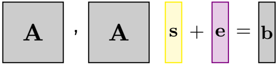
- $\mathsf{PartDec}(\mathsf{sk}_i, \mathsf{ct})$:
  - Sample $e_i \leftarrow D_{flood}$
  - Output $d_i = \mathbf{u}\mathbf{s}_i + e_i$



- $\mathsf{Combine}(d_1, \ldots, d_n)$:
  - $d = \sum_{i=1}^{n} d_i$
  - If $v - d$ is closer to 0 than to $q/2$, output $m' = 0$
  - Else output $m' = 1$

Full-Threshold PKE from LWE [BD10]

- KGen(1^λ):
  - $\mathbf{A} \leftarrow$ Unif$(\mathbb{Z}_q^{n\times n})$ and $\mathbf{s}, \mathbf{e} \leftarrow \chi^n$
  - $\mathbf{b} = \mathbf{A}\mathbf{s} + \mathbf{e} \bmod q$
  - $\mathbf{s} = \sum_{i=1}^{n} \mathbf{s}_i$
  - Output $\mathsf{sk}_i = \mathbf{s}_i$ and $\mathsf{pk} = (\mathbf{A}, \mathbf{b})$



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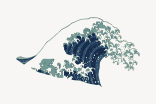

- Combine$(d_1, \ldots, d_n)$:
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Correctness:

$$v - \sum_{i=1}^{n} \mathbf{u}\mathbf{s}_i + e_i = v - \mathbf{u}\sum_{i=1}^{n} \mathbf{s}_i + e_i = v - \mathbf{u}\mathbf{s} + \sum_{i=1}^{n} e_i$$

$$= \underbrace{\mathbf{r}\mathbf{e} + f' - \mathbf{f}\mathbf{s} + \sum_{i=1}^{n} e_i}_{*} + \lfloor q/2 \rfloor m$$

Decryption succeeds if $|*| < q/8$

Put under the carpet for today ...

⚠ It is non-trivial to go from full-threshold to arbitrary threshold PKE
if you are working with lattices ;-)

n-out-of-n threshold

$\sum_{i=1}^{n} e_i$

t-out-of-n threshold

$\sum_{i \in S} \lambda_i e_i$

still needs to be small

? There are solutions, but not very efficient for large n.

Partial Decryption Security

Two worlds:

- Real: $e_{\mathsf{ct}} = \mathbf{re} + f' - \mathbf{fs}$ and $e_{flood} = \sum_i e_i$
- Simulated: only $e_{flood} = \sum_i e_i$

How close are they? [BD10] measures with statistical distance Δ

$$\Delta(\mathsf{Real}, \mathsf{Sim}) \leq \Delta(e_{flood} + e_{\mathsf{ct}}, e_{flood}) \leq \mathsf{negl}(\lambda)$$

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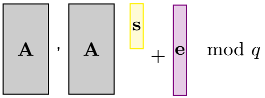
- LWE-based constructions: $\|e_{flood}\| \sim$ LWE modulus q and $\|e_{\mathsf{ct}}\| \sim$ LWE noise $\mathbf{e}$, thus super-polynomial modulus-noise ratio
  - Larger parameters
  - Easier problem



Partial Decryption Security

Idea: change the measure! [BLR+18]

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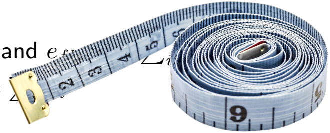

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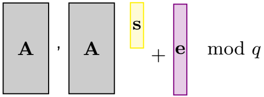

Improved Noise Flooding via Rényi Divergence 1/2

Let P, Q be discrete probability distributions

In [BD10]: Statistical Distance $\Delta(P,Q) = \frac{1}{2}\sum_{x \in \mathrm{Supp}(P)} |P(x) - Q(x)|$

In [BS23]: Rényi Divergence

$$\mathsf{RD}(P,Q) = \sum_{\substack{x \in \mathrm{Supp}(P) \\ \subset \mathrm{Supp}(Q)}} \frac{P(x)^2}{Q(x)}$$

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Both fulfill the **probability preservation property** for an event E:

[BD10]:	$P(E)$	$\leq$	$\Delta(P,Q) + Q(E)$	(additive)
Our work:	$P(E)^2$	$\leq$	$\mathrm{RD}(P,Q) \cdot Q(E)$	(multiplicative)

- $Q(E)$ negligible $\Rightarrow$ $P(E)$ negligible
- $\Delta(P,Q) \stackrel{!}{=}$ negligible and $\mathrm{RD}(P,Q) \stackrel{!}{=}$ **constant**

Improved Noise Flooding via Rényi Divergence 2/2

Two worlds:

- Real: e_{ct} and e_{flood}
- Simulated: only e_{flood}

How close are they?

$$\Delta(\mathsf{Real}, \mathsf{Sim}) \leq \Delta(e_{flood} + e_{\mathsf{ct}}, e_{flood}) \leq \mathsf{negl}(\lambda)$$

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Disadvantage:

1) Rényi divergence depends on the number of issued partial decryptions
   $\rightarrow$ from simulation-based to game-based security notion
2) Works well with search problems, not so well with decision problems

Zooming out - leakage on secret key

Two worlds:

- Real: $f(\mathsf{sk})$ and e_{flood} — f some function
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- Threshold decryption: $f(\mathsf{sk})$ is the ciphertext noise [BS23]
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> “ We don't yet understand very well when which approach is **optimal** ”

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Wrap-Up

Hopefully you have now a rough idea:

- Part 1: *What lattices are!*
- Part 2: *What lattice problems are!*
- Part 3: *What lattice-based cryptography is!*
- Part 4: *What particular challenges are!*

Any questions or interested in my research?

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¡Muchas Gracias!

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