# Post-Quantum Cryptography from Lattices

*Cryptographie post-quantique à base de réseaux euclidiens*

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https://katinkabou.github.io/

# Merci à Michel pour l'invitation au Sénégal !

# Overview

Questions we try to answer today:

- *What is post-quantum cryptography?*
- *What are lattice problems?*
- *What is lattice-based cryptography?*
- *What are some (of my) current challenges?*

References:

- Crash Course Spring 2025
  https://katinkabou.github.io/LatticeClub2025.html
- The Lattice Club
  https://thelatticeclub.com/

# Part 1:

## *Post-Quantum Cryptography*

# Cryptography

👉 The word **cryptography** is composed of the two ancient Greek words *kryptos* (hidden) and *graphein* (to write). Its goal is to provide **secure communication**.

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- Digital Signatures

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- Encryption
- Digital Signatures
- Zero-Knowledge Proofs
- Fully-Homomorphic Encryption

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 5 | 3 | | | 7 | | | | |
| 6 | | | 1 | 9 | 5 | | | |
| | 9 | 8 | | | | | 6 | |
| 8 | | | | 6 | | | | 3 |
| 4 | | | 8 | | 3 | | | 1 |
| 7 | | | | 2 | | | | 6 |
| | 6 | | | | | 2 | 8 | |
| | | | 4 | 1 | 9 | | | 5 |
| | | | | 8 | | | 7 | 9 |

# Cryptography is everywhere!



# Security Reductions

Security of cryptographic scheme ⟵ Mathematical problem

e.g. an adversary cannot find the secret key

e.g. it is difficult to factor a number $N$ which is the product of two large primes $N = p \cdot q$

👉 The security in cryptography is based on presumably hard mathematical problems.

# Current Security Paradigm

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Currently used problems:

- Discrete logarithm
- Factoring

  Given $N$, find $p, q$ such that $N = p \cdot q$

$^{\star}$Shor, *Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer*, SIAM Journal of Computations 1997

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- ~~Discrete logarithm~~
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  Given $N$, find $p, q$ such that $N = p \cdot q$

⚠ $\exists$ poly-time quantum algorithm [Sho97]*

Quantum-resistant candidates:

- Codes
- Lattices $\Rightarrow$ today's focus
- Isogenies
- Multivariate systems
- ?

*Shor, *Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer*, SIAM Journal of Computations 1997

# Post-Quantum Cryptography

☞ Post-quantum cryptography denotes schemes which plausibly resist attacks by quantum computers.

# US National Institute of Standards and Technology (NIST) Project

- 2016: start of NIST's post-quantum cryptography project*
- 2022+25: selection of 5 schemes, 3 of them relying on lattice problems

Public Key Encryption:

- Kyber
- HQC

Digital Signature:

- Dilithium
- Falcon
- SPHINCS+

Lattice-based cryptography plays a leading role in designing post-quantum cryptography.

* https://csrc.nist.gov/projects/post-quantum-cryptography

# Lattices are more than just post-quantum!

Example: Fully-Homomorphic Encryption

- Securely outsource data and do analysis on the encrypted data
- Very powerful
- Only known from lattices so far

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BUT: Lattices also bring new challenges! More later . . .

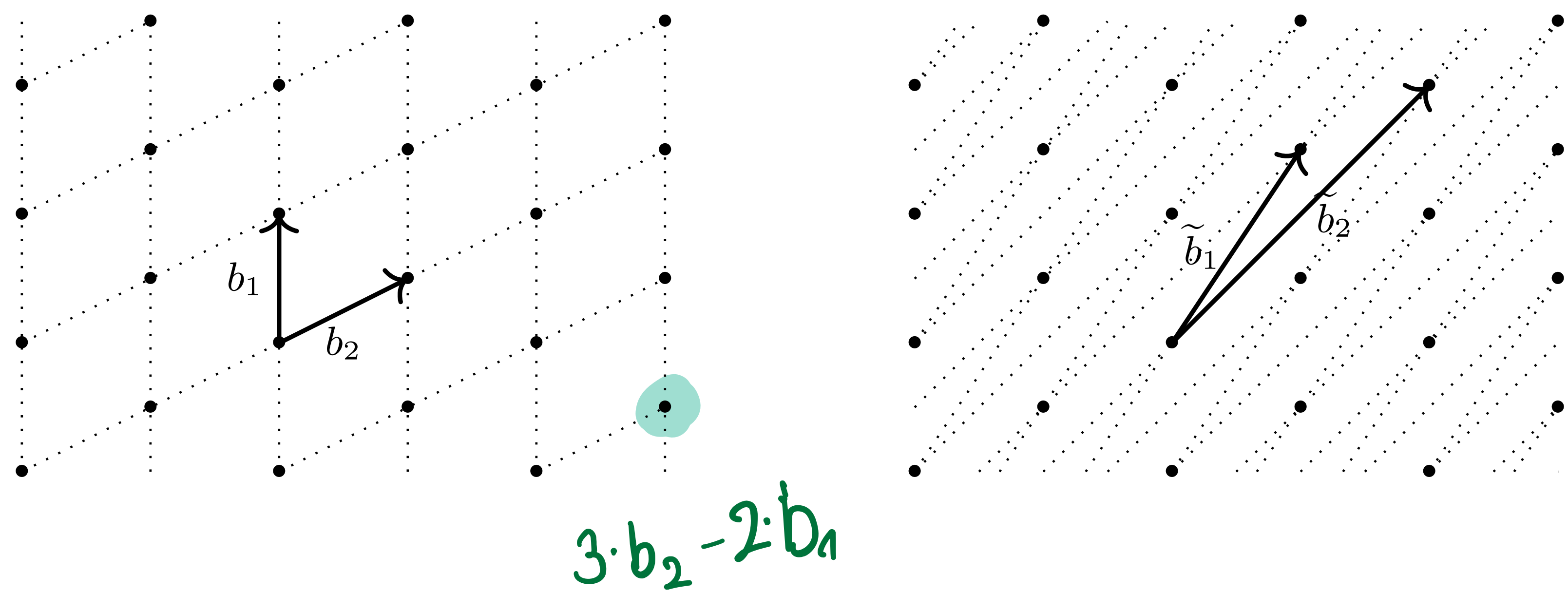
# Part 2:

# *Euclidean Lattice Problems*

# Euclidean Lattices

Let $\mathbf{B} = (\mathbf{b}_i)_{i=1,\ldots,n}$ be a set of linearly independent vectors over $\mathbb{R}$, defining the lattice

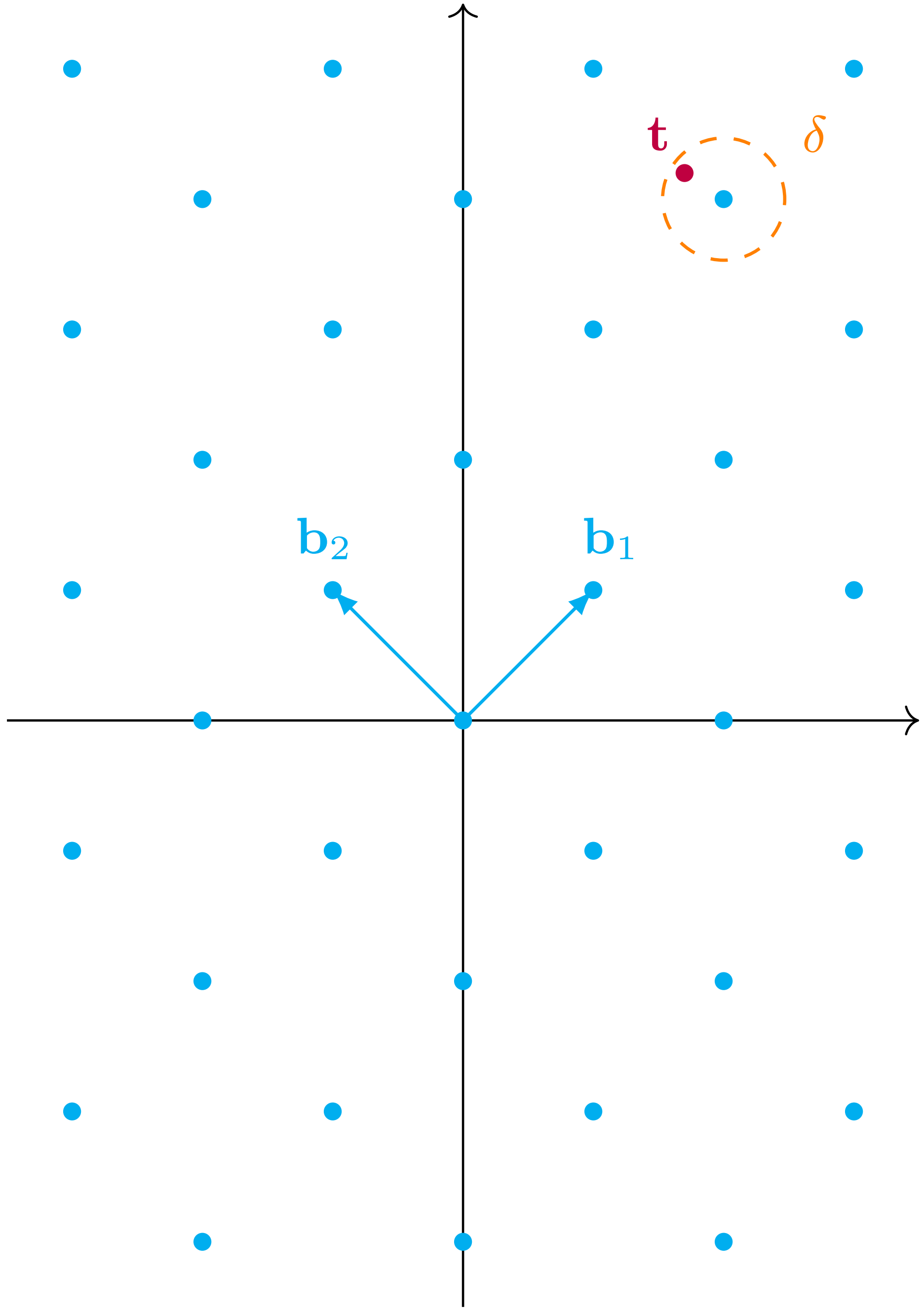
$$\Lambda(\mathbf{B}) = \left\{ \sum_{i=1}^{n} z_i \mathbf{b}_i : z_i \in \mathbb{Z} \right\}.$$



# Bounded Distance Decoding

Given a lattice $\Lambda$ and a target $\mathbf{t}$ such that

$$\mathrm{dist}(\Lambda, \mathbf{t}) \leq \delta.$$



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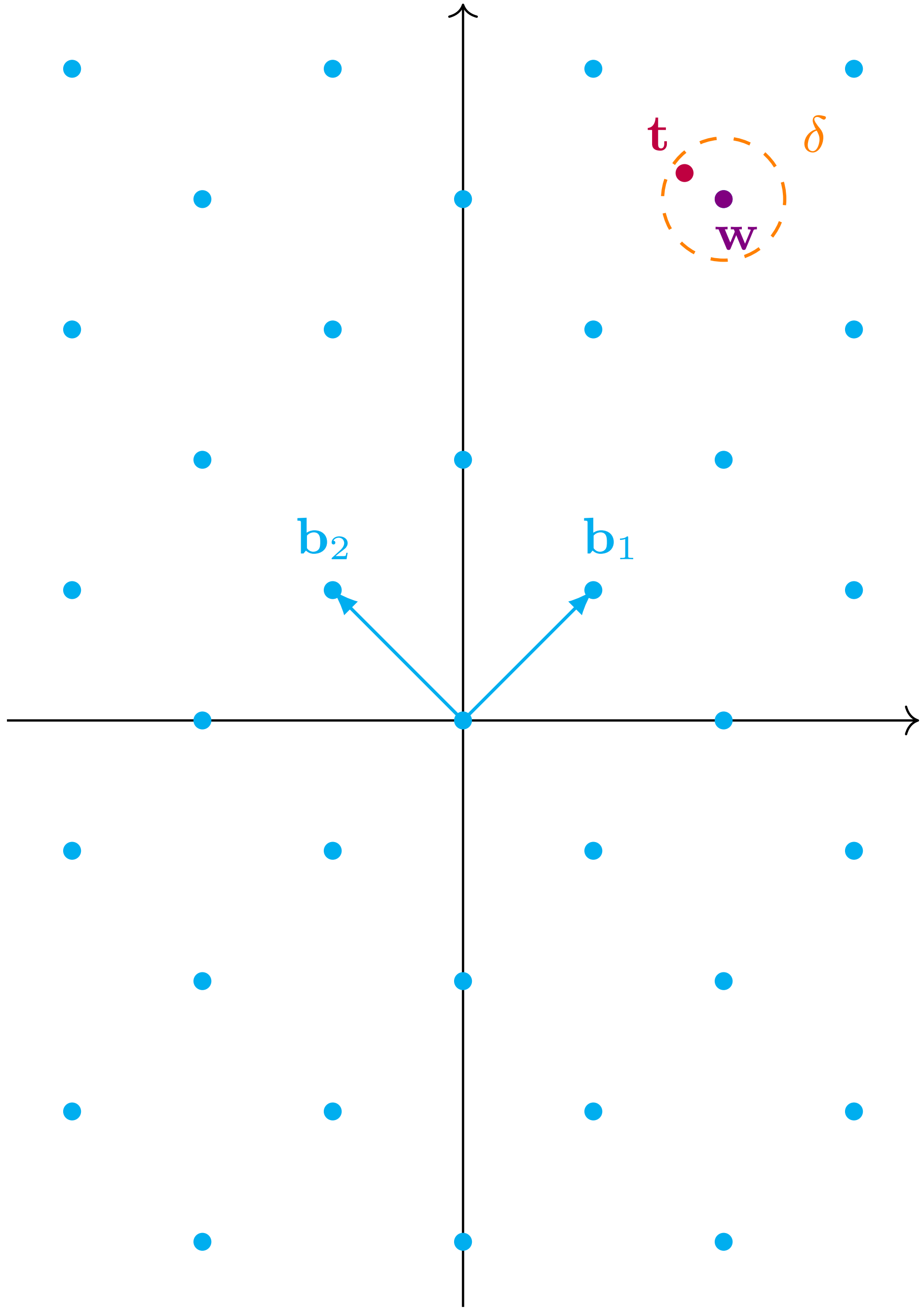

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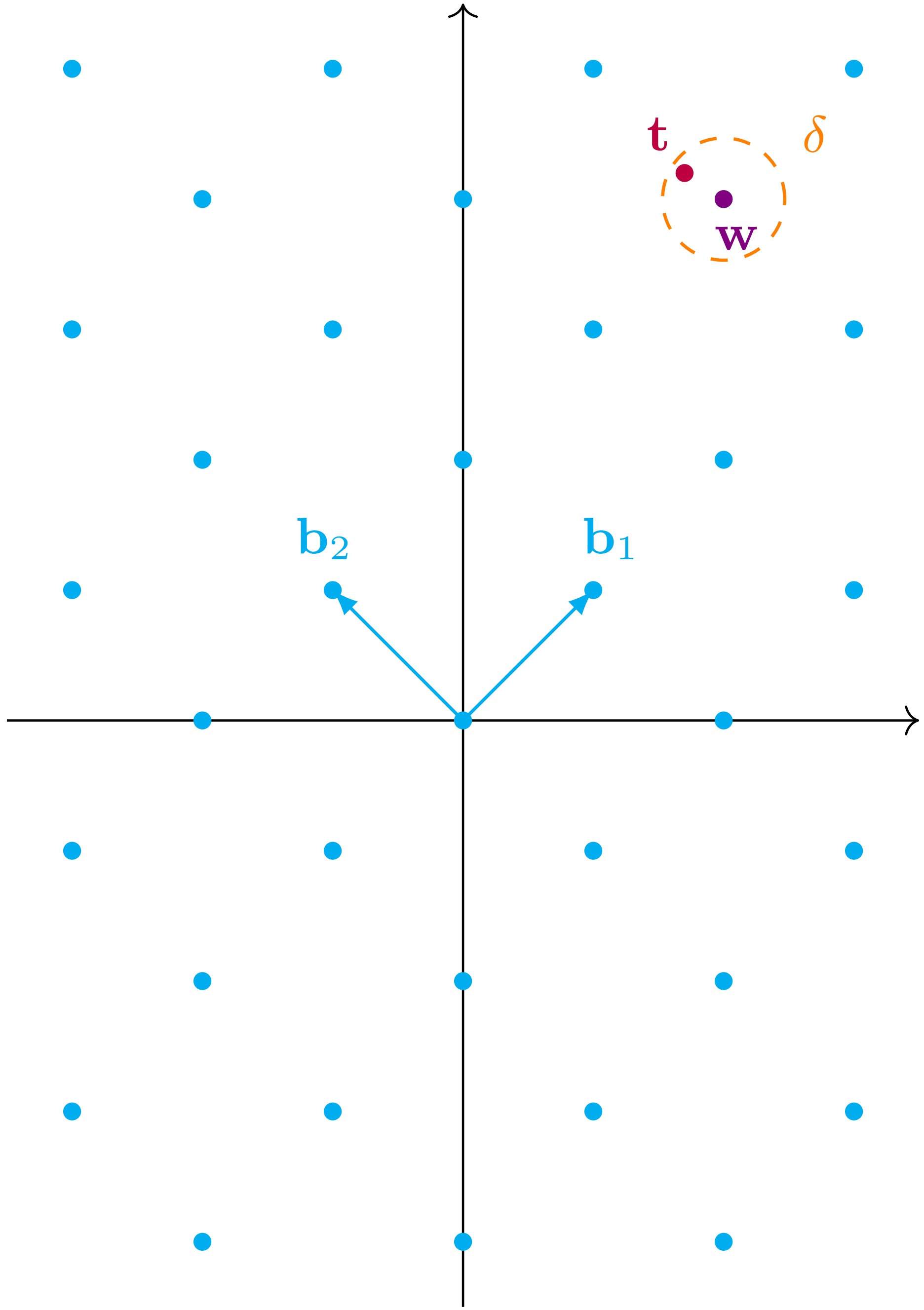
The complexity of BDD increases with the lattice dimension and promised radius $\delta$.

**Conjecture:**
There is no polynomial-time classical or quantum algorithm that solves BDD **for all** lattices to within polynomial factors.

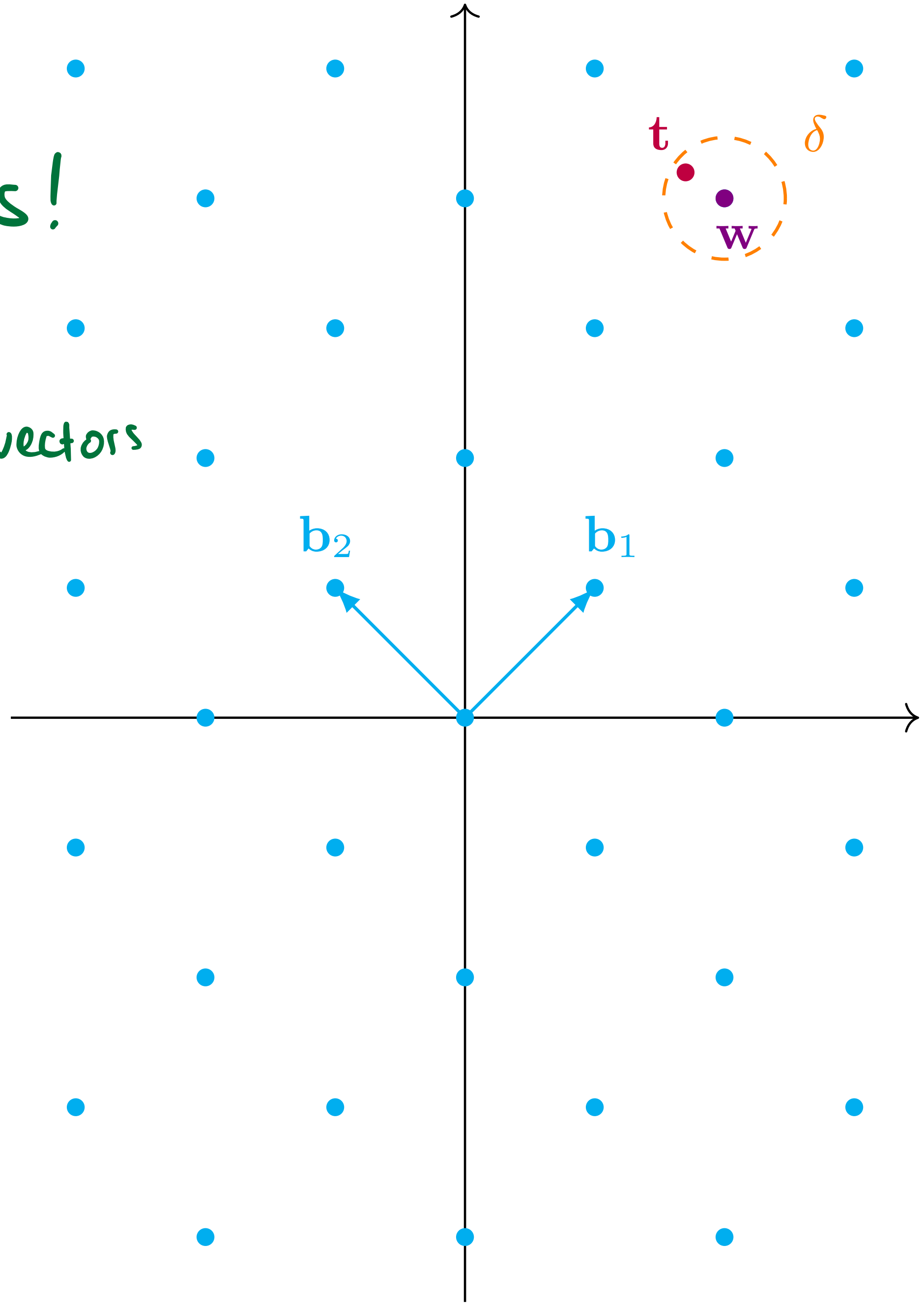


# Bounded Distance Decoding

But: BDD might be easy to solve for some lattices!

Example: $\Lambda = \mathbb{Z}^n$ generated by unit vectors
⇝ simply round

How do we sample "hard" instances?

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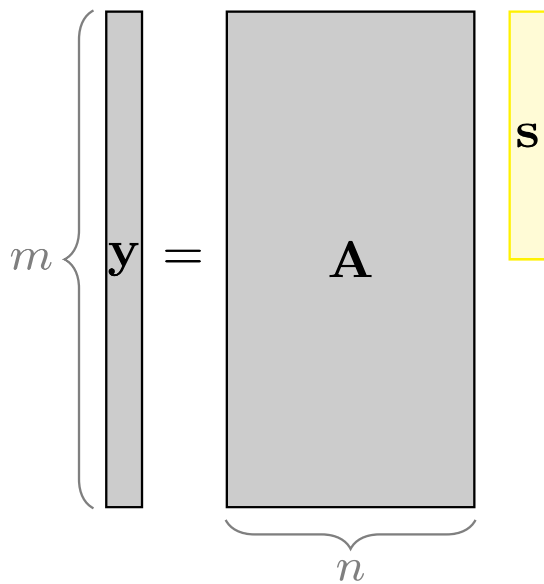
# A family of random lattices

$\mathbb{Z}_q$ = Integers modulo $q$
$\cong \{0, \ldots, q-1\}$

- Let $\mathbb{Z}_q$ be a finite field
- Sample $\mathbf{A} \in \mathbb{Z}_q^{m \times n}$ uniformly at random
- Define the lattice $\Lambda_q(\mathbf{A}) = \{\mathbf{y} \in \mathbb{Z}^m : \mathbf{y} = \mathbf{A}\mathbf{s} \bmod q$ for some $\mathbf{s} \in \mathbb{Z}^n\}$



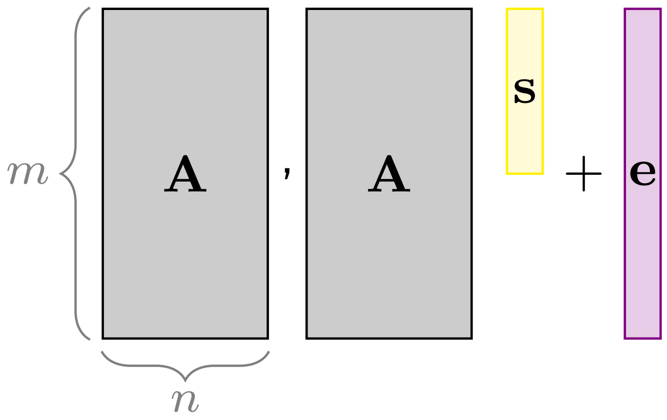

over $\mathbb{Z}_q$

# Learning With Errors

Given a matrix $\mathbf{A} \in \mathbb{Z}_q^{m \times n}$ sampled uniformly at random.

Given a vector $\mathbf{b} \in \mathbb{Z}_q^m$, where $\mathbf{b} = \mathbf{A}\mathbf{s} + \mathbf{e} \bmod q$ for

- secret $\mathbf{s} \in \mathbb{Z}^n$ sampled from distribution $D_s$ and
- noise/error $\mathbf{e} \in \mathbb{Z}^m$ sampled from distribution $D_e$ such that $\|\mathbf{e}\|_2 \leq \delta \ll q$.



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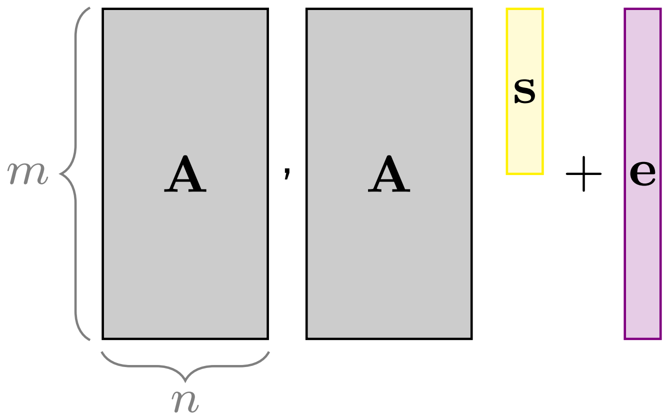


**Search** learning with errors (S-LWE) asks to find $\mathbf{s}$.

**Decision** learning with errors (D-LWE) asks to distinguish $(\mathbf{A}, \mathbf{b})$ from the uniform distribution over $\mathbb{Z}_q^{m \times n} \times \mathbb{Z}_q^m$.

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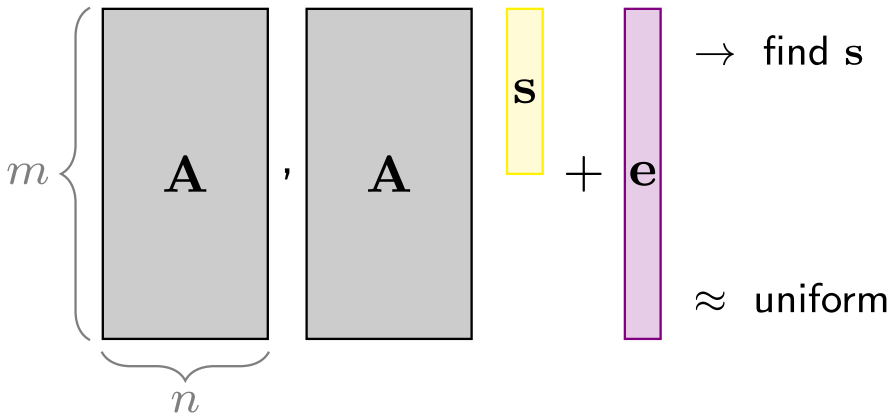


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⚠ The present noise makes S-LWE a hard problem.

⚠ The norm restriction on $\mathbf{e}$ makes D-LWE a hard problem!

# Learning With Errors

- Introduced by [Reg05]*
- Most important hardness assumption in lattice-based cryptography
- $=$ Bounded distance decoding in random lattices $\Lambda_q(\mathbf{A})$
- $\approx$ Solving random noisy linear equations over finite fields

* Regev, *On lattices, learning with errors, random linear codes, and cryptography*, STOC'05

# Example Parameters for Learning With Errors

- LWE is flexible $\rightarrow$ good for constructions
- LWE is parametrized by multiple parameters $\rightarrow$ various choices possible
  - Integers $m, n$ and $q$
  - Distribution of error $D_e$
  - Distribution of secret $D_s$

# Example Parameters for Learning With Errors

- LWE is flexible $\rightarrow$ good for constructions
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For simplicity, $D_e$ and $D_s$ bounded uniform distribution with infinity norm bound $\delta$.

| $n, m$ | $q$ | $\delta$ | security bits |
|---|---|---|---|
| 512 | 3329 | 3 | 118 |
| 768 | 3329 | 2 | 183 |
| 1024 | 3329 | 3 | 256 |

million of years all humans together

# Part 3:

## *How to build encryption schemes from lattices*

# Reminder: Encryption

An encryption scheme $\Pi = (\mathsf{KGen}, \mathsf{Enc}, \mathsf{Dec})$ consists of three algorithms:

- $\mathsf{KGen} \to \mathsf{sk}$
- $\mathsf{Enc}(\mathsf{sk}, m) \to \mathsf{ct}$
- $\mathsf{Dec}(\mathsf{sk}, \mathsf{ct}) = m'$

**Correctness:** $\mathsf{Dec}(\mathsf{sk}, \mathsf{Enc}(\mathsf{sk}, m)) = m$ during an honest execution

**Security:** $\mathsf{Enc}(\mathsf{sk}, m_0)$ is indistinguishable from $\mathsf{Enc}(\mathsf{sk}, m_1)$

# Encryption from LWE

Let $D_s$ and $D_e$ be secret and error distributions and $\mathbb{Z}_q$ be a finite field.

KGen:

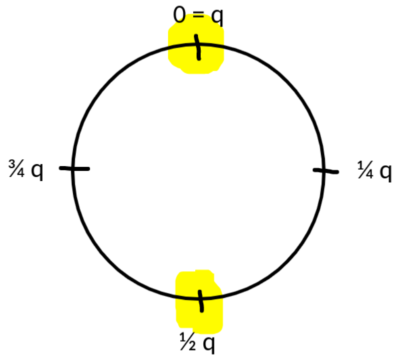
- Output $\mathbf{s} \leftarrow D_s$

Enc($\mathbf{s}, m \in \{0,1\}^n$):

- $\mathbf{A} \leftarrow \text{Unif}(\mathbb{Z}_q^{n \times n})$
- $\mathbf{e} \leftarrow D_e$
- $\mathbf{u} = \mathbf{A}\mathbf{s} + \mathbf{e} + \lfloor q/2 \rfloor \cdot m \bmod q$
- Output $(\mathbf{A}, \mathbf{u})$

Dec($\mathbf{s}, \mathbf{A}, \mathbf{u}$):

- For every coefficient of $\mathbf{u} - \mathbf{A}\mathbf{s}$:
- If closer to $0$ than to $q/2$, output $0$
- Else output $1$



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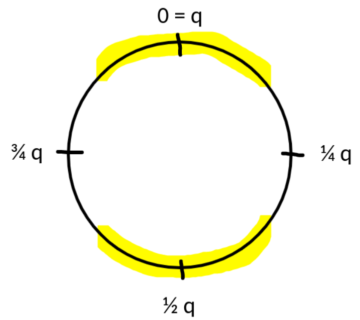


**Correctness:**

$$\begin{aligned} \mathbf{u} - \mathbf{A}\mathbf{s} &= \mathbf{A}\mathbf{s} + \mathbf{e} + \lfloor q/2 \rfloor \cdot m - \mathbf{A}\mathbf{s} \\ &= \mathbf{e} + \lfloor q/2 \rfloor m \end{aligned}$$

Decryption succeeds if $\|\mathbf{e}\|_\infty < q/8$

# Encryption from LWE 2/2

Let $D_s$ and $D_e$ be secret and error distributions and $\mathbb{Z}_q$ be a finite field.

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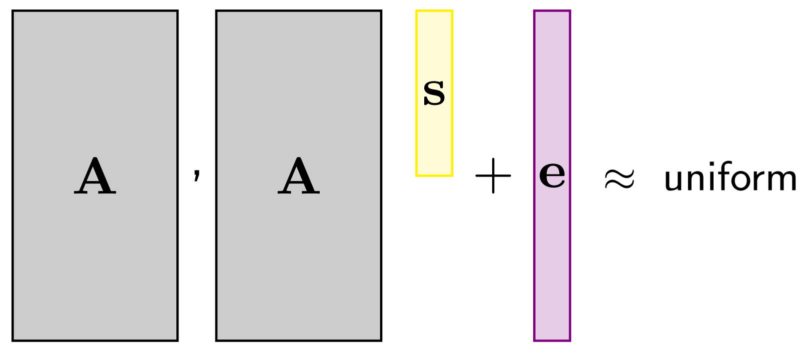


**Security:**

- Assume hardness of decision Learning with Errors (D-LWE)
- $\mathbf{A}\mathbf{s} + \mathbf{e} + \lfloor q/2 \rceil m_0 \approx$ uniform $+ \lfloor q/2 \rceil m_0 \approx$ uniform $+ \lfloor q/2 \rceil m_1 \approx \mathbf{A}\mathbf{s} + \mathbf{e} + \lfloor q/2 \rceil m_1$
- Encryption of $m_0$ indistinguishable from encryption of $m_1$

# Part 4:

## *Some (of my) current challenges*

# Challenges from Encryption

KGen:

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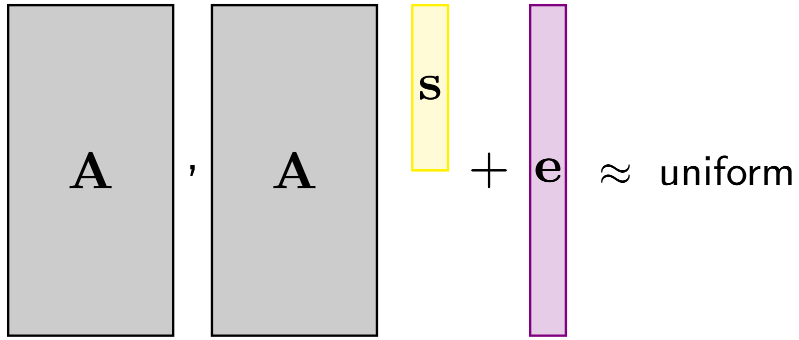
- Difficult to distribute calculation among multiple people [BS23]*
- Linearly split $\mathbf{s} = \mathbf{s}_1 + \mathbf{s}_2 \Rightarrow \mathbf{A}\mathbf{s}_1 + \mathbf{A}\mathbf{s}_2 = \mathbf{A}\mathbf{s}$
- How to "split" the non-linear rounding step ?

* Boudgoust and Scholl, *Simple Threshold (Fully Homomorphic) Encryption From LWE With Polynomial Modulus*, Asiacrpyt'23

# Challenges from LWE

- Many options for secret distribution $D_s$ and error distribution $D_e$
- For different choices same hardness?
- Goal: show that only the min-entropy (and norm bound) matter
- Theoretical answer: [BJTW25]*
- What if we change structure of $\mathbf{A}$?




*Boudgoust, Jeudy, Tairi, Wen, *Hardness of M-LWE with General Distributions and Applications to Leaky Variants* IACR ePrint 2025/1472

# Wrap-Up

Hopefully you have now a rough idea:

- *What post-quantum cryptography is:*
  Cryptography assumed to be secure against quantum computers
- *What lattice problems are:*
  Learning with Errors (LWE): Noisly random linear equations in finite fields
- *How to build cryptography from lattices:*
  Encryption from LWE
- *What new challenges come with lattices:*
  Distributing computations & LWE choices

Any questions or interested in my research?

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Jerejef !

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